

E₀-SEMIGROUPS AND *q*-PURITY

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ABSTRACT. An E₀-semigroup is called *q*-pure if it is a CP-flow and its set of flow subordinates is totally ordered by subordination. The range rank of a positive boundary weight map is the dimension of the range of its dual map. Let \mathfrak{K} be a separable Hilbert space. We describe all *q*-pure E₀-semigroups of type II₀ which arise from boundary weight maps with range rank one over \mathfrak{K} . We also prove that no *q*-pure E₀-semigroups of type II₀ arise from boundary weight maps with range rank two over \mathfrak{K} . In the case when \mathfrak{K} is finite-dimensional, we provide a criterion to determine if two boundary weight maps of range rank one over \mathfrak{K} give rise to cocycle conjugate *q*-pure E₀-semigroups.

1. INTRODUCTION

Let $\mathfrak{B}(\mathfrak{H})$ be algebra of all bounded operators on a separable Hilbert space \mathfrak{H} . A CP-semigroup $\alpha = \{\alpha_t\}$ acting on $\mathfrak{B}(\mathfrak{H})$ is a continuous one-parameter semigroup of contractive completely positive maps which is continuous in the point-strong operator topology. When α_t is in addition an endomorphism for every $t > 0$, then α is called an E-semigroup, and if furthermore α_t unital, then α is an E₀-semigroup. We recommend the monograph by Arveson [2] as an excellent introduction to the theory of E₀-semigroups, and we will make use of its terminology in the remainder.

Given α and β two CP-semigroups acting on $\mathfrak{B}(\mathfrak{H})$, we say that β is a subordinate of α if $\alpha_t - \beta_t$ is a completely positive map for all $t > 0$. Given an E₀-semigroup α , let $\mathfrak{S}(\alpha)$ be the set of all its CP-semigroup subordinates, endowed with the partial order given by subordination. Let also $\mathfrak{E}(\alpha)$ be the subset of all E-semigroup subordinates of α . Both partially ordered sets are easily seen to be cocycle conjugacy invariants of α . The set $\mathfrak{S}(\alpha)$ was first studied by Bhat [3], who characterized it completely in the type I case, with the help of the quantum stochastic calculus in the sense of Hudson-Parthasarathy [7]. Liebscher [11] has further described an alternative presentation for $\mathfrak{E}(\alpha)$ in terms of subproduct systems arising from certain measure types with partial order given by absolute continuity.

We would like to improve our understanding of the class of E₀-semigroups α such that $\mathfrak{S}(\alpha)$ is as small as possible. When α is an E₀-semigroup and additionally it is a CP-flow, then we will call it *q*-pure if and only if its set of CP-flow subordinates is totally ordered by subordination. See Section 4 for a detailed discussion. It seems to us that *q*-pure E₀-semigroups will become important objects in the classification theory of E₀-semigroups.

The class of *q*-pure E₀-semigroups was first studied by Powers [14]. Recall that Powers [15] has proven that any spatial E₀-semigroup can be constructed (up to cocycle conjugacy) by choosing an appropriate *q*-weight map over a separable Hilbert space \mathfrak{K} . A *q*-weight map over \mathfrak{K} is a boundary weight map from the predual of $\mathfrak{B}(\mathfrak{K})$ to certain weights on a (non-closed) *-subalgebra $\mathfrak{A}(\mathfrak{H}) \subseteq \mathfrak{B}(\mathfrak{K} \otimes L^2(0, \infty))$ satisfying the *q*-positivity conditions. Powers [14] completely described the *q*-weight maps over $\mathfrak{K} = \mathbb{C}$ which give rise to *q*-pure E₀-semigroups. Later Jankowski [8] analyzed a class of E₀-semigroups of type II₀ arising from boundary weight doubles (ϕ, ν) where $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a *q*-positive linear map and ν is an unbounded boundary weight over $L^2(0, \infty)$. Jankowski characterized those boundary weight doubles giving rise to *q*-pure E₀-semigroups when the map ϕ either is invertible or has rank one and in addition ν has the particular form $\nu(T) = (f, (I - \Lambda)^{-1/2} T (I - \Lambda)^{-1/2} f)$ where $\Lambda \in B(L^2(0, \infty))$ is the multiplication operator by e^{-x} , $T \in \mathfrak{A}(\mathfrak{H})$ and $f \in L^2(0, \infty)$.

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In this paper we generalize the results of Powers [14] and Jankowski [8] to the class \mathfrak{C} of all E_0 -semigroups of type II_0 arising from q -weight maps with range rank one over a separable Hilbert space \mathfrak{H} . The range rank of a q -weight map $\omega : \mathfrak{B}(\mathfrak{H})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ is defined to be the dimension of the range of the dual map (see Definition 2.12 and the discussion preceding it for the details). In particular, the class \mathfrak{C} contains all q -weights considered earlier by Powers [14] as well as those considered by Jankowski [8] with ϕ rank one. In fact, we obtain an effective description of all q -weight maps in the class \mathfrak{C} which are q -pure (see Theorem 5.7). We also prove that if \mathfrak{H} is finite-dimensional, then a q -corner between unbounded range rank one q -weight maps must have range rank one (see Theorem 5.8). And we describe one criterion to determine if a q -corner exists (see Theorem 5.14) between q -pure boundary weight maps of range rank one over \mathfrak{H} finite-dimensional. The latter results form an important first step for the classification theory of q -pure E_0 -semigroups. See Section 5.

We also show that if a q -weight map over a separable Hilbert space \mathfrak{H} has range rank two, then it cannot give rise to an E_0 -semigroup of type II_0 that is q -pure (Corollary 6.4). In fact, when $\dim \mathfrak{H} = 2$, we prove further that a q -weight map can only give rise to a q -pure E_0 -semigroup of type II_0 if it has range rank 1 or 4. This generalizes a result of Jankowski [9]. This comprises Section 6, which is the last section of the article.

We describe the remaining sections of the paper. In Section 2 we review the basic terminology and results necessary for the paper. In Section 3 we introduce the concept of *boundary expectations*, which has proved useful and convenient for the analysis of certain properties of q -weight maps with range rank one or two.

2. PRELIMINARIES

In this article, we will consider only Hilbert spaces that are separable unless stated otherwise. We will also denote the inner product of the Hilbert space by (\cdot, \cdot) and it will be taken to be conjugate linear in the *first* entry.

2.1. E_0 -semigroups and CP-flows.

Definition 2.1. Let \mathfrak{H} be a separable Hilbert space. We say a family $\alpha = \{\alpha_t\}_{t \geq 0}$ of normal completely positive contractions of $\mathfrak{B}(\mathfrak{H})$ into itself is a *CP-semigroup* acting on $\mathfrak{B}(\mathfrak{H})$ if:

- (i) $\alpha_s \circ \alpha_t = \alpha_{s+t}$ for all $s, t \geq 0$ and $\alpha_0(A) = A$ for all $A \in \mathfrak{B}(\mathfrak{H})$;
- (ii) For each $f, g \in \mathfrak{H}$ and $A \in \mathfrak{B}(\mathfrak{H})$, the inner product $(f, \alpha_t(A)g)$ is continuous in t ;

If $\alpha_t(I) = I$ for all $t \geq 0$, then α is called a *unital* CP-semigroup. When α is a unital CP-semigroup and in addition the map α_t is an endomorphism for every $t \geq 0$, then α is called an *E_0 -semigroup*.

We have two notions of equivalence for E_0 -semigroups:

Definition 2.2. An E_0 -semigroup α acting on $\mathfrak{B}(\mathfrak{H}_1)$ is *conjugate* to an E_0 -semigroup β acting on $\mathfrak{B}(\mathfrak{H}_2)$ if there exists a $*$ -isomorphism θ from $\mathfrak{B}(\mathfrak{H}_1)$ onto $\mathfrak{B}(\mathfrak{H}_2)$ such that $\theta \circ \alpha_t = \beta_t \circ \theta$ for all $t \geq 0$.

A strongly continuous family of contractions $\mathcal{W} = \{W_t\}_{t \geq 0}$ acting on \mathfrak{H}_2 is called a *contractive β -cocycle* if $W_t \beta_t(W_s) = W_{t+s}$ for all $t, s \geq 0$. A contractive β -cocycle W_t is said to be a *local cocycle* if for all $A \in \mathfrak{B}(\mathfrak{H}_2)$ and $t \geq 0$, $W_t \beta_t(A) = \beta_t(A)W_t$.

We say α and β are *cocycle conjugate* if there exists a unitary β -cocycle $\{W_t\}_{t \geq 0}$ such that the E_0 -semigroup acting on $\mathfrak{B}(\mathfrak{H}_2)$ given by $\beta'_t(A) = W_t \beta_t(A) W_t^*$ for all $A \in \mathfrak{B}(\mathfrak{H}_2)$ and $t \geq 0$ is conjugate to α .

Let \mathfrak{H} be a separable Hilbert space. We will always denote by $\{S_t\}_{t \geq 0}$ the right shift semigroup on $\mathfrak{H} \otimes L^2(0, \infty)$ (which we identify with the space of \mathfrak{H} -valued measurable functions on $(0, \infty)$ which are square integrable):

$$(S_t f)(x) = \begin{cases} f(x-t), & x > t; \\ 0, & x \leq t. \end{cases}$$

We will also denote by $E(t, \infty) = S_t S_t^*$ for all $t \geq 0$, and $E(t, s) = E(t, \infty) - E(s, \infty)$ for all $0 \leq t < s < \infty$.

Definition 2.3. A CP-semigroup α acting on $\mathfrak{B}(\mathfrak{H} \otimes L^2(0, \infty))$ is called a *CP-flow* over \mathfrak{H} if $\alpha_t(A)S_t = S_t A$ for all $A \in \mathfrak{B}(\mathfrak{H} \otimes L^2(0, \infty))$ and $t \geq 0$.

A dilation of a unital CP-semigroup α acting on $\mathfrak{B}(\mathfrak{K})$ is a pair (α^d, W) , where α^d is an E₀-semigroup acting on $\mathfrak{B}(\mathfrak{H})$ and $W : \mathfrak{K} \rightarrow \mathfrak{H}$ is an isometry such that $\alpha_t^d(WW^*) \geq WW^*$ for $t > 0$ and furthermore

$$\alpha_t(A) = W^* \alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{K})$ and $t \geq 0$. The dilation is said to be *minimal* if the span of the vectors

$$\alpha_{t_1}^d(WA_1W^*)\alpha_{t_2}^d(WA_2W^*)\cdots\alpha_{t_n}^d(WA_nW^*)Wf$$

for $f \in K, A_i \in \mathfrak{B}(\mathfrak{K}), i = 1, \dots, n, n \in \mathbb{N}$ is dense in \mathfrak{H} . This definition of minimality is due to Arveson (see [2] for a detailed discussion regarding dilations of CP-semigroups). We will often suppress the isometry W , and refer to a minimal dilation α^d instead of (α^d, W) .

Theorem 2.4 (Bhat's dilation theorem). *Every unital CP-semigroup has a minimal dilation which is unique up to conjugacy.*

The following addendum by Powers (Lemma 4.50 of [15]) further clarifies the situation for CP-flows.

Theorem 2.5. *Every unital CP-flow α has a minimal dilation α^d which is also a CP-flow. We call α^d the minimal flow dilation of the unital CP-flow.*

Given two CP-flows α and β over \mathfrak{K} , we will say that α *dominates* β or that β is a *subordinate* of α if for all $t \geq 0$, the map $\alpha_t - \beta_t$ is completely positive. We will often denote this relationship by $\alpha \geq \beta$. Powers [15] has described a useful criterion for determining whether two CP-flows have minimal dilations that are cocycle conjugate in terms of the next definition.

Definition 2.6. Let α and β be CP-flows over \mathfrak{K}_1 and \mathfrak{K}_2 , respectively. For $j = 1, 2$, let $\mathfrak{H}_j = \mathfrak{K}_j \otimes L^2(0, \infty)$ and let $S_t^{(j)}$ denote the right shift on \mathfrak{H}_j . Let $\gamma = \{\gamma_t : t \geq 0\}$ be a family of maps from $\mathfrak{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ into itself and define for each $t > 0$, $\gamma_t^* : \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2) \rightarrow \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ by $\gamma_t^*(C) = [\gamma_t(C^*)]^*$ for all $C \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. We say that γ is a *flow corner* from α to β if the maps

$$\Theta_t \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha_t(A) & \gamma_t(B) \\ \gamma_t^*(C) & \beta_t(D) \end{bmatrix}$$

define a CP-flow $\Theta = \{\Theta_t : t \geq 0\}$ over $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ with respect to the shift $S_t^{(1)} \oplus S_t^{(2)}$. Note that γ is a flow corner from α to β if and only if γ^* is a flow corner from β to α .

A flow corner γ is called a *hyper-maximal flow corner* if every subordinate CP-flow Θ' of Θ of the form

$$\Theta'_t \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha'_t(A) & \gamma_t(B) \\ \gamma_t^*(C) & \beta'_t(D) \end{bmatrix}$$

for $t \geq 0$ must satisfy $\alpha'_t = \alpha_t$ and $\beta'_t = \beta_t$ for all $t \geq 0$.

Theorem 2.7. *Suppose α and β are unital CP-flows over \mathfrak{K}_1 and \mathfrak{K}_2 and α^d and β^d are their minimal dilations to E₀-semigroups. Suppose γ is a hyper maximal flow corner from α to β . Then α^d and β^d are cocycle conjugate. Conversely, if α^d is a type II₀ and α^d and β^d are cocycle conjugate, then there is a hyper maximal flow corner from α to β .*

2.2. Boundary weight maps. For the remainder of this section, let \mathfrak{K} be a fixed separable Hilbert space (not necessarily infinite-dimensional) and let $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$.

Define $\Lambda : \mathfrak{B}(\mathfrak{K}) \rightarrow \mathfrak{B}(\mathfrak{H})$ by

$$(\Lambda(A)f)(x) = e^{-x}Af(x)$$

and let $\mathfrak{A}(\mathfrak{H})$ be the algebra

$$\mathfrak{A}(\mathfrak{H}) = [I - \Lambda(I_{\mathfrak{K}})]^{\frac{1}{2}}\mathfrak{B}(\mathfrak{H})[I - \Lambda(I_{\mathfrak{K}})]^{\frac{1}{2}}.$$

Definition 2.8. We say that a linear functional $\mu : \mathfrak{A}(\mathfrak{H}) \rightarrow \mathbb{C}$ is a *boundary weight*, denoted $\mu \in \mathfrak{A}(\mathfrak{H})_*$, if the functional ℓ defined on $\mathfrak{B}(\mathfrak{H})$ by

$$\ell(A) = \mu\left([I - \Lambda(I_{\mathfrak{K}})]^{\frac{1}{2}}A[I - \Lambda(I_{\mathfrak{K}})]^{\frac{1}{2}}\right)$$

is a normal bounded linear functional. The boundary weight μ is called *bounded* if there exists $C > 0$ such that $|\mu(T)| \leq C\|T\|$ for all $T \in \mathfrak{A}(\mathfrak{H})$. Otherwise, μ is called *unbounded*.

A linear map from $\mathfrak{B}(\mathfrak{K})_*$ to $\mathfrak{A}(\mathfrak{H})_*$ will be called a *boundary weight map*.

Boundary weights were first defined in [15] (Definition 4.16), where their relationship to CP-flows was explored in depth. For an additional discussion of boundary weights and their properties, we refer the reader to Definition 1.10 of [12] and its subsequent remarks.

Given a normal map $\phi : \mathfrak{B}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K})$, we will denote by $\hat{\phi} : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{B}(\mathfrak{H})_*$ the predual map satisfying $\rho(\phi(A)) = (\hat{\phi}(\rho))(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$.

Define $\Gamma : \mathfrak{B}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{H})$ by the weak* integral

$$(1) \quad \Gamma(A) = \int_0^\infty e^{-t} S_t A S_t^* dt.$$

The following records facts that are implicit in the proof of Theorem 4.17 in Powers [15] (for a proof, see Proposition 2.11 of [10]):

Proposition 2.9. *Let $\mu \in \mathfrak{A}(\mathfrak{H})_*$ be a boundary weight. We have that for all $T \in \mathfrak{A}(\mathfrak{H})$,*

$$\mu(T) = \lim_{x \rightarrow 0^+} \mu(E(x, \infty) T E(x, \infty)).$$

In particular $\mu = \mu'$ if and only if for all $x > 0$ and $T \in E(x, \infty) \mathfrak{B}(\mathfrak{H}) E(x, \infty)$, we have that $\mu(T) = \mu'(T)$. Furthermore, given $x > 0$ and $T \in E(x, \infty) \mathfrak{B}(\mathfrak{H}) E(x, \infty)$,

$$(2) \quad \mu(T) = \lim_{y \rightarrow x^+} \frac{1}{y - x} \hat{\Gamma}(\mu) \left(T - e^{x-y} S_{y-x} T S_{y-x}^* \right).$$

If α is a CP-flow over \mathfrak{K} , we define its resolvent by the weak* integral

$$(3) \quad R_\alpha(A) = \int_0^\infty e^{-t} \alpha_t(A) dt$$

defined for $A \in \mathfrak{B}(\mathfrak{H})$. Powers [15] proved that there exists a completely positive boundary weight map $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ such that

$$(4) \quad \hat{R}_\alpha(\eta) = \hat{\Gamma}(\omega(\hat{\Lambda}\eta) + \eta)$$

and $\omega(\rho)(I - \Lambda(I_{\mathfrak{K}})) \leq \rho(I_{\mathfrak{K}})$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ positive. Such a boundary weight map is uniquely determined by (4) in combination with Proposition 2.9, and in fact for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$, $x > 0$ and $T \in E(x, \infty) \mathfrak{B}(\mathfrak{H}) E(x, \infty)$,

$$(5) \quad \omega(\rho)(T) = \lim_{y \rightarrow x^+} \frac{1}{y - x} (\hat{R}_\alpha - \hat{\Gamma})(\eta)(T - e^{x-y} S_{y-x} T S_{y-x}^*),$$

where $\eta \in \mathfrak{B}(\mathfrak{H})_*$ is any normal functional such that $\rho = \hat{\Lambda}(\eta)$. Such a functional exists since Λ is isometric hence $\hat{\Lambda}$ is onto.

The map ω is called *the boundary weight map associated to α* .

The following result, which is a compilation of Theorems 4.17, 4.23, and 4.27 of [15], describes the converse relationship between boundary weight maps and CP-flows.

Theorem 2.10. *Let $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ be a completely positive map satisfying $\omega(\rho)(I - \Lambda(I_{\mathfrak{K}})) \leq \rho(I_{\mathfrak{K}})$ for all positive ρ . Let $\{S_t\}_{t \geq 0}$ be the right shift semigroup acting on \mathfrak{H} . For each $t > 0$, define the truncated boundary weight map $\omega|_t : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{B}(\mathfrak{H})_*$ by*

$$\omega|_t(\rho)(A) = \omega(\rho)(E(t, \infty) A E(t, \infty))$$

If for every $t > 0$, the map $(I + \hat{\Lambda}\omega|_t)$ is invertible and furthermore the map

$$(6) \quad \hat{\pi}_t := \omega|_t(I + \hat{\Lambda}\omega|_t)^{-1}$$

is a completely positive contraction from $\mathfrak{B}(\mathfrak{K})_$ into $\mathfrak{B}(\mathfrak{H})_*$, then ω is the boundary weight map associated to a CP-flow over \mathfrak{K} . The CP-flow is unital if and only if $\omega(\rho)(I - \Lambda(I_{\mathfrak{K}})) = \rho(I_{\mathfrak{K}})$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$.*

Definition 2.11. Let $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ be a completely positive boundary weight map satisfying $\omega(\rho)(I - \Lambda(I_{\mathfrak{K}})) \leq \rho(I_{\mathfrak{K}})$ for all positive ρ . If for every $t > 0$ the map $\hat{\pi}_t$ as defined in the statement of Theorem 2.10 exists and it is a completely positive contraction, then ω is called a *q-weight map over \mathfrak{K}* . In that case, the family $\pi_t^\#$ (for $t > 0$) of completely positive normal contractions from $\mathfrak{B}(\mathfrak{H})$ to

$\mathfrak{B}(\mathfrak{K})$ is called the *generalized boundary representation* associated to ω , or alternatively to the CP-flow associated to ω . For every $t > 0$, we have

$$(7) \quad \omega|_t = \hat{\pi}_t(I - \hat{\Lambda}\hat{\pi}_t)^{-1}.$$

We say a q -weight map ω is *unital* if it induces a unital CP-flow. By Theorem 2.10, ω is unital if and only if $\omega(\rho)(I - \Lambda(I_{\mathfrak{K}})) = \rho(I_{\mathfrak{K}})$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$.

We will say that a q -weight $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ is *bounded* if for every $\rho \in \mathfrak{B}(\mathfrak{K})_*$ the linear functional $\omega(\rho)$ extends to a σ -weakly continuous linear functional on $\mathfrak{B}(\mathfrak{H})$.

We note that, in order to check that $\pi_t^\#$ is a completely positive contraction for all $t > 0$, it suffices to check for small t in the sense that if $\pi_s^\#$ is a completely positive contraction for some $s > 0$, then for all $t > s$, $\pi_t^\#$ is automatically a completely positive contraction.

If $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ is a completely positive boundary weight map, then we have a well-defined completely positive map $\tilde{\omega} : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{B}(\mathfrak{H})_*$ given by

$$\tilde{\omega}(\rho)(A) = \omega(\rho)((I - \Lambda)^{\frac{1}{2}}A(I - \Lambda)^{\frac{1}{2}}) \quad \forall \rho \in \mathfrak{B}(\mathfrak{K})_*, \forall A \in \mathfrak{B}(\mathfrak{H}).$$

By an argument analogous to the proof of continuity of positive linear functionals C^* -algebras, the positivity of $\tilde{\omega}$ implies that it is bounded.

Since $\tilde{\omega} : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{B}(\mathfrak{H})_*$ is a bounded linear map, it induces a normal dual map $\tilde{\omega}' : \mathfrak{B}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K})$ satisfying

$$\rho(\tilde{\omega}'(A)) = \tilde{\omega}(\rho)(A) \quad \forall A \in \mathfrak{B}(\mathfrak{H}).$$

Observe that $\tilde{\omega}'$ is completely positive since $\tilde{\omega}$ has that property.

Since the map $\mathfrak{B}(\mathfrak{H}) \rightarrow \mathfrak{A}(\mathfrak{H})$ given by $A \mapsto (I - \Lambda)^{\frac{1}{2}}A(I - \Lambda)^{\frac{1}{2}}$ is one-to-one and onto, there is a unique linear map $\tilde{\omega} : \mathfrak{A}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K})$ satisfying

$$(8) \quad \tilde{\omega}((I - \Lambda)^{\frac{1}{2}}A(I - \Lambda)^{\frac{1}{2}}) = \tilde{\omega}'(A) \quad \forall A \in \mathfrak{B}(\mathfrak{H}).$$

Definition 2.12. Let $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ be a completely positive boundary weight map. We define the *dualized boundary weight map* $\tilde{\omega} : \mathfrak{A}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K})$ to be the unique map satisfying equation (8) or, alternatively,

$$\rho(\tilde{\omega}(B)) = \tilde{\omega}(\rho)(B) \quad \forall \rho \in \mathfrak{B}(\mathfrak{K})_*, \forall B \in \mathfrak{A}(\mathfrak{H}).$$

Similarly, for every $t > 0$, there exists a unique normal map $\tilde{\omega}|_t : \mathfrak{B}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K})$ such that

$$\rho(\tilde{\omega}|_t(A)) = \omega|_t(\rho)(A)$$

for every $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{B}(\mathfrak{H})$ or, alternatively, $\tilde{\omega}|_t(A) = \tilde{\omega}(E(t, \infty)AE(t, \infty))$ for all $t > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$.

We will say that a q -weight map $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ has *finite range rank* if $\text{Range}(\tilde{\omega}) \subseteq \mathfrak{B}(\mathfrak{K})$ is finite-dimensional. In this case, we will say that the *range rank* of ω is the dimension of $\text{Range}(\tilde{\omega})$. Of course, if $\dim \mathfrak{K} < \infty$, then ω automatically has finite range rank.

In the next result proven by Powers [15] we recall the criterion for subordination in terms of the generalized boundary representation.

Theorem 2.13. Let α and β be CP-flows acting on $\mathfrak{B}(\mathfrak{H})$ with generalized boundary representations $\pi_t^\#$ and $\xi_t^\#$, respectively. Then $\alpha \geq \beta$ if and only if $\pi_t^\# - \xi_t^\#$ is completely positive for all $t > 0$. Also if $\pi_s^\# \geq \xi_s^\#$ for some $s > 0$, then $\pi_t^\# \geq \xi_t^\#$ for all $t \geq s$, so one only has to check for a sequence $(t_n)_{n \in \mathbb{N}}$ tending to zero.

We can deduce from Theorem 2.13 that there is a bijective correspondence between CP-flows and q -weight maps: Let α and β be CP-flows, with associated q -weight maps ω and η and generalized boundary representations $\{\pi_t^\#\}_{t>0}$ and $\{\xi_t^\#\}_{t>0}$, respectively. By Theorem 2.13, $\alpha = \beta$ if and only if $\pi_t^\# = \xi_t^\#$ for every $t > 0$. By equations (6) and (7), this holds if and only if $\omega|_t = \eta|_t$ for all $t > 0$. By Proposition 2.9, we have $\omega|_t = \eta|_t$ for all $t > 0$ if and only if $\omega = \eta$. Therefore, $\alpha = \beta$ if and only if $\omega = \eta$.

The index of the E_0 -semigroup induced by a CP-flow turns out to be the rank of an associated map which is called its normal spine:

Theorem 2.14. *Let α be a CP-flow over \mathfrak{K} with generalized boundary representation $\pi_t^\#$. Then for every $A \in \cup_{t>0} S_t \mathfrak{B}(\mathfrak{H}) S_t^*$, we have that $\pi_t^\#(A)$ converges σ -strongly to an element denoted by $\pi_0^\#(A)$. Furthermore, this map extends uniquely to a map $\pi_0^\# : \mathfrak{B}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K})$, called the normal spine of α , which is a σ -weakly continuous completely positive contraction. The index of α^d is equal to the rank of $\pi_0^\#$ as a completely positive map.*

We note that in the particular case when α is a CP-flow which induces an E_0 -semigroup type II_0 , then it follows that its normal spine $\pi_0^\# = 0$.

2.3. Generalized Schur maps. Recall that a map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is said to be a *Schur map* if there exists a matrix $Q = (q_{ij}) \in M_n(\mathbb{C})$ such that

$$\phi((x_{ij})) = (q_{ij}x_{ij})$$

In this section we review the concept and notation associated with generalized Schur maps introduced in [10], and which will be used in the remainder of the paper.

For each $i = 1, 2, \dots, n$, let \mathfrak{K}_i and \mathfrak{H}_i be Hilbert spaces, and let $\mathfrak{K} = \bigoplus_{i=1}^n \mathfrak{K}_i$ and $\mathfrak{H} = \bigoplus_{i=1}^n \mathfrak{H}_i$. Let for $i = 1, \dots, n$, $V_i : \mathfrak{K}_i \rightarrow \mathfrak{K}$ and $W_i : \mathfrak{H}_i \rightarrow \mathfrak{H}$ be the canonical isometries. Given operators $A \in \mathfrak{B}(\mathfrak{K})$ and $B \in \mathfrak{B}(\mathfrak{H})$, and for $i, j = 1, 2, \dots, n$ given operators $X \in \mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)$, $Z \in \mathfrak{B}(\mathfrak{H}_j, \mathfrak{H}_i)$, we define

$$\begin{aligned} A_{ij} &= V_i^* A V_j \in \mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i) & X^{ij} &= V_i X V_j^* \in \mathfrak{B}(\mathfrak{K}) \\ B_{ij} &= W_i^* B W_j \in \mathfrak{B}(\mathfrak{H}_j, \mathfrak{H}_i) & Z^{ij} &= W_i Z W_j^* \in \mathfrak{B}(\mathfrak{H}) \end{aligned}$$

In particular,

$$(X^{ij})_{rs} = \delta_{ir} \delta_{js} X.$$

Given a subalgebra \mathfrak{A} of $\mathfrak{B}(\mathfrak{H})$, and for each $i, j = 1, 2, \dots, n$, let $\mathfrak{A}_{ij} = W_i^* \mathfrak{A} W_j$. Suppose that for all $i, j = 1, 2, \dots, n$,

$$(9) \quad W_i \mathfrak{A}_{ij} W_j^* \subseteq \mathfrak{A}.$$

Given a linear map $\phi : \mathfrak{A} \rightarrow \mathfrak{B}(\mathfrak{K})$, for each $i, j = 1, 2, \dots, n$ we define the linear map $\phi_{ij} : \mathfrak{A}_{ij} \rightarrow \mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)$ given by

$$\phi_{ij}(X) = [\phi(X^{ij})]_{ij}$$

We say that ϕ is a *generalized Schur map* with respect to the decompositions $\bigoplus_{i=1}^n \mathfrak{K}_i$ and $\bigoplus_{i=1}^n \mathfrak{H}_i$ if for all $A \in \mathfrak{A}$,

$$[\phi(A)]_{ij} = \phi_{ij}(A_{ij}).$$

In particular, if ϕ is a generalized Schur map and if $X \in \mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)$, then

$$\phi(X^{ij}) = [\phi_{ij}(X)]^{ij}.$$

A similar definition applies to maps from $\mathfrak{B}(\mathfrak{K})_*$ to the algebraic dual \mathfrak{A}' . If $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $\eta \in \mathfrak{A}'$, we define for each $i, j = 1, 2, \dots, n$ the linear functionals $\rho_{ij} \in \mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)'$ and $\eta_{ij} \in \mathfrak{A}'_{ij}$ given by

$$\rho_{ij}(X) = \rho(X^{ij}), \quad \eta_{ij}(Z) = \eta(Z^{ij}),$$

for all $X \in \mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)$ and $Z \in \mathfrak{A}_{ij}$. For each $\mu \in \mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)'$, we define $\mu^{ij} \in \mathfrak{B}(\mathfrak{K})'$ given by

$$\mu^{ij}(A) = \mu(A_{ij}).$$

Given a map $\Psi : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}'$ and $i, j = 1, 2, \dots, n$, we define $\Psi_{ij} : \mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)' \rightarrow \mathfrak{A}'_{ij}$ by

$$\Psi_{ij}(\mu) = [\Psi(\mu^{ij})]_{ij}.$$

We say that $\Psi : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}'$ is a *generalized Schur map with respect to the decompositions $\bigoplus_{i=1}^n \mathfrak{K}_i$ and $\bigoplus_{i=1}^n \mathfrak{H}_i$* if

$$[\Psi(\rho)]_{ij} = \Psi_{ij}(\rho_{ij}).$$

We observe that if Ψ is a generalized Schur map and $\rho \in \mathfrak{B}(\mathfrak{K})_*$, then $\Psi([\rho_{ij}]^{ij}) = [\Psi_{ij}(\rho_{ij})]^{ij}$.

2.4. Powers weights and boundary weight doubles. For a moment, let us examine the case when ω is a q -weight map over \mathbb{C} . Then ω is determined by its value $\omega_1 := \omega(1)$, and it induces a CP-flow α over \mathbb{C} if and only if ω_1 is a positive boundary weight and $\omega_1(I - \Lambda) \leq 1$. In that case, the CP-flow α is unital if and only if $\omega_1(I - \Lambda) = 1$, and therefore dilates to an E_0 -semigroup α^d .

Since all the key properties of ω are determined by the single boundary weight ω_1 in the special case $\mathfrak{K} = \mathbb{C}$, we will write ω instead of ω_1 .

Results from [15] show that α^d is of type I if ω_1 is bounded and of type II₀ if ω_1 is unbounded. This leads to the following definition.

Definition 2.15. A boundary weight $\nu \in \mathfrak{A}(L^2(0, \infty))_*$ is called a *Powers weight* if ν is positive and $\nu(I - \Lambda) = 1$. We say that a Powers weight ν is *type I* if it is bounded and *type II* if it is unbounded.

If ν is a Powers weight, then it has the form:

$$\nu\left((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}}\right) = \sum_{i=1}^k (f_i, A f_i)$$

for some mutually orthogonal nonzero L^2 -functions $\{f_i\}_{i=1}^k$ ($k \in \mathbb{N} \cup \{\infty\}$) with $\sum_{i=1}^k \|f_i\|^2 = 1$. We note that if ν is a type II Powers weight, then for the weights ν_t defined by $\nu_t(A) = \nu(E(t, \infty) A E(t, \infty))$ for $A \in B(L^2(0, \infty))$ and $t > 0$, both $\nu_t(I)$ and $\nu_t(\Lambda)$ approach infinity as $t \rightarrow 0+$.

In [14], Powers defined q -corners and hyper-maximal q -corners, and determined necessary and sufficient conditions for cocycle conjugacy between E_0 -semigroups arising from type II Powers weights. In the following, we will generalize several of the definitions and results obtained in [14], such as Definition 3.11 and Theorems 3.9, 3.10 and 3.14.

2.5. Comparison theory for q -weight maps. Suppose that $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ is a q -weight map which induces a CP-flow α with generalized boundary representation $\pi_t^\#$. It will be important to describe the subordinates of α in terms of ω .

Definition 2.16. A q -weight map η with associated generalized boundary representation $\xi_t^\#$ is called a q -subordinate of ω if for all $t > 0$ we have $\xi_t^\# \leq \pi_t^\#$. We will denote this relation by $\eta \leq_q \omega$.

In view of Theorem 2.13, it is clear that q -subordination of q -weight maps is equivalent to subordination for the associated CP-flows. We will also make use of the following fact.

Proposition 2.17. Let $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ be a q -weight map and let η be a q -subordinate of ω . Then for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and positive $T \in \mathfrak{A}(\mathfrak{H})$,

$$(10) \quad \omega(\rho)(T) \geq \eta(\rho)(T).$$

Furthermore, if $\omega(\rho)(I - \Lambda) = \eta(\rho)(I - \Lambda)$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$, then $\omega = \eta$.

Proof. Let $\pi_t^\#$ and $\xi_t^\#$ be the generalized boundary representations for ω and η respectively. By Theorem 2.13, we have that for all $t > 0$, $\pi_t^\# - \xi_t^\#$ is completely positive. Therefore,

$$\omega|_t = \hat{\pi}_t \sum_{n=0}^{\infty} (\hat{\Lambda} \hat{\pi}_t)^n \geq \hat{\xi}_t \sum_{n=0}^{\infty} (\hat{\Lambda} \hat{\xi}_t)^n = \eta|_t$$

for all $t > 0$ (the series converge, at every positive functional ρ in the sense of weights). The inequality is in the completely positive sense. Therefore, we have that for all $T \in \mathfrak{A}(\mathfrak{H})$ positive, and for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$, by Proposition 2.9,

$$\omega(\rho)(T) - \eta(\rho)(T) = \lim_{t \rightarrow 0+} \omega|_t(\rho)(T) - \eta|_t(\rho)(T) \geq 0.$$

Thus we have that $\omega \geq \eta$ (in the positive sense). Therefore, we have that $\hat{\omega} - \hat{\eta}$ is positive as a map from $\mathfrak{B}(\mathfrak{K})_*$ to $\mathfrak{B}(\mathfrak{H})_*$, or alternatively, $\hat{\omega}(\rho) - \hat{\eta}(\rho)$ is a positive normal functional for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Now notice that for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$,

$$\hat{\omega}(\rho)(I) = \omega(\rho)(I - \Lambda) = \eta(\rho)(I - \Lambda) = \hat{\eta}(\rho)(I).$$

Therefore, it follows that for every positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$, $\hat{\omega}(\rho) = \hat{\eta}(\rho)$. Now by considering linear combinations, $\hat{\omega} = \hat{\eta}$, hence $\omega = \eta$. \square

Now suppose that ω_i is a unital q -weight map for $i = 1, 2$, so ω_i induces a unital CP-flow α_i . By Theorem 2.4, α_i has a minimal dilation to an E_0 -semigroup α_i^d which is unique up to conjugacy. A fundamental question to ask is whether α_1^d and α_2^d are cocycle conjugate. Theorem 2.7 gives us a partial answer in terms of flow corners from α_1 to α_2 . In this section, we translate this description in terms of boundary weight maps. We will use the following notation. If \mathfrak{K}_i is a separable Hilbert space and $\mathfrak{H}_i = \mathfrak{K}_i \otimes L^2(0, \infty)$ for $i = 1, 2$, then we denote by $\mathfrak{A}(\mathfrak{H}_i, \mathfrak{H}_j)$ the vector space

$$\mathfrak{A}(\mathfrak{H}_i, \mathfrak{H}_j) = [I - \Lambda(I_{\mathfrak{K}_j})]^{1/2} \mathfrak{B}(\mathfrak{H}_i, \mathfrak{H}_j) [I - \Lambda(I_{\mathfrak{K}_i})]^{1/2}.$$

Let $\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$ and $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. We define $\mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)_*$ to be the closed subspace of $\mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)'$ given by

$$\mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)_* = \{\rho \in \mathfrak{B}(\mathfrak{K}_j, \mathfrak{K}_i)' : \rho^{ij} \in \mathfrak{B}(\mathfrak{K})_*\},$$

and similarly we define the vector space

$$\mathfrak{A}(\mathfrak{H}_j, \mathfrak{H}_i)_* = \{\nu \in \mathfrak{A}(\mathfrak{H}_j, \mathfrak{H}_i)' : \nu^{ij} \in \mathfrak{A}(\mathfrak{H})_*\}.$$

For the sake of clarity, we will frequently write generalized Schur maps in matrix form. For example, if $\rho \in \mathfrak{B}(\mathfrak{K}_1 \oplus \mathfrak{K}_2)'$, so $\rho = \sum_{i,j=1}^2 (\rho_{ij})^{ij}$, we write it as

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix},$$

and if ω is a boundary weight map over $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ which is also a generalized Schur map, we denote it by

$$\omega(\rho) = \begin{pmatrix} \omega_{11}(\rho_{11}) & \omega_{12}(\rho_{12}) \\ \omega_{21}(\rho_{21}) & \omega_{22}(\rho_{22}) \end{pmatrix},$$

or write ω in the more abbreviated form

$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}.$$

Definition 2.18. For $i = 1, 2$, let \mathfrak{K}_i be a separable Hilbert space, and let $\mathfrak{H}_i = \mathfrak{K}_i \otimes L^2(0, \infty)$. Suppose $\mu : \mathfrak{B}(\mathfrak{K}_1)_* \rightarrow \mathfrak{A}(\mathfrak{H}_1)_*$ and $\eta : \mathfrak{B}(\mathfrak{K}_2)_* \rightarrow \mathfrak{A}(\mathfrak{H}_2)_*$ are q -weight maps. We say that a map ℓ from $\mathfrak{B}(\mathfrak{K}_2, \mathfrak{K}_1)_* \rightarrow \mathfrak{A}(\mathfrak{H}_2, \mathfrak{H}_1)_*$ is a q -corner from μ to η if $\omega : \mathfrak{B}(\mathfrak{K}_1 \oplus \mathfrak{K}_2)_* \rightarrow \mathfrak{A}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)_*$ defined by

$$\omega(\rho) = \begin{pmatrix} \mu(\rho_{11}) & \ell(\rho_{12}) \\ \ell^*(\rho_{21}) & \eta(\rho_{22}) \end{pmatrix} \quad \forall \rho \in \mathfrak{B}(\mathfrak{K}_1 \oplus \mathfrak{K}_2)_*$$

is a q -weight map over $\mathfrak{K}_1 \oplus \mathfrak{K}_2$. We say ℓ is a *hyper-maximal q -corner* from μ to η if, whenever

$$\omega \geq_q \begin{pmatrix} \mu' & \ell \\ \ell^* & \eta' \end{pmatrix} \geq_q 0,$$

we have $\mu = \mu'$ and $\eta = \eta'$.

The following result has a straightforward proof using the techniques of generalized Schur maps introduced in [10], which we omit.

Theorem 2.19. Suppose α and β are unital CP-flows over \mathfrak{K}_1 and \mathfrak{K}_2 with boundary weight maps μ and η , respectively. If there is a hyper-maximal q -corner from μ to η , then α^d and β^d are cocycle conjugate. Conversely, if α^d is a type II_0 E_0 -semigroup and α^d and β^d are cocycle conjugate, then there is a hypermaximal q -corner from μ to η .

3. BOUNDARY EXPECTATIONS

Definition 3.1. Let ω be a q -weight map over a separable Hilbert space \mathfrak{K} . We will say that a map $L : \mathfrak{B}(\mathfrak{K}) \rightarrow \mathfrak{B}(\mathfrak{K})$ is a *boundary expectation* corresponding to ω (or alternatively the CP-flow it induces) if it satisfies the following properties:

- (i) L is completely positive;
- (ii) $L \circ \check{\omega} = \check{\omega}$;
- (iii) $\text{Range}(L) = \text{Range}(\check{\omega})$;
- (iv) $L^2 = L$ and $\|L\| = 1$.

We note that we do not know if a boundary expectation always exists, but in general even when one exists it need not be unique. However, as we establish in the following, a boundary expectation always exists when ω is a q -weight map with finite range rank whose normal spine is zero.

Let \mathfrak{K} be a separable Hilbert space. Recall that the relative BW-topology on the set $\mathfrak{C}(\mathfrak{B}(\mathfrak{K})) = \{\phi : \mathfrak{B}(\mathfrak{K}) \rightarrow \mathfrak{B}(\mathfrak{K}) : \|\phi\| \leq 1\}$ is determined by requiring that a net ϕ_λ converges to ϕ if and only if for all $x \in \mathfrak{B}(\mathfrak{K})$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$, we have $\rho(\phi_\lambda(x)) \rightarrow \rho(\phi(x))$. Furthermore, $\mathfrak{C}(\mathfrak{B}(\mathfrak{K}))$ is compact in the relative BW-topology (see [1] for details).

Theorem 3.2. *Let \mathfrak{K} be a separable Hilbert space. Let ω be a non-zero q -weight map over \mathfrak{K} with finite range rank, and let $\{\Pi_t^\#\}_{t>0}$ be the corresponding generalized boundary representation. Let us consider $(\Pi_t^\# \circ \Lambda)_{t>0}$ as a net with respect to the directed $J = (0, \infty)$ directed by $t \lesssim s$ if and only if $t \geq s$. If $\Pi_0^\# \equiv 0$, then the net $(\Pi_t^\# \circ \Lambda)_{t \in J}$ in $\mathfrak{C}(\mathfrak{B}(\mathfrak{K}))$ has at least one cluster point L in the relative BW-topology, and every such cluster point is a boundary expectation for ω .*

Proof. Since $\Pi_t^\# \circ \Lambda$ is a completely positive contraction of $\mathfrak{B}(\mathfrak{K})$ for each $t > 0$, we have that the net is inside the set $\mathfrak{C}(\mathfrak{B}(\mathfrak{K}))$ which is compact in the relative BW-topology. Therefore, there exists a subnet $\Pi_{t_\mu}^\# \circ \Lambda$ which converges to a map L in the BW-topology. We now show that L has properties (i) through (iv) of Definition 3.1. Property (i) follows trivially from the fact that the space of completely positive maps in $\mathfrak{C}(\mathfrak{B}(\mathfrak{K}))$ is closed in the BW-topology.

Let $s > 0$. We claim that

$$(11) \quad \lim_{t \rightarrow 0^+} \|(I + \check{\omega}|_t \circ \Lambda)^{-1} \circ \check{\omega}|_s\| = 0.$$

For this, we first note that if $t < s$ and $A \in \mathfrak{B}(\mathfrak{H})$, then

$$(12) \quad \check{\omega}|_s(A) = \check{\omega}(E(s, \infty)AE(s, \infty)) = \check{\omega}|_t(E(s, \infty)AE(s, \infty)).$$

Since $\Pi_0^\#$ is the zero map and $E(s, \infty) \in U_s \mathfrak{B}(\mathfrak{H}) U_s^*$, it follows that $\Pi_t^\#(E(s, \infty)) \rightarrow 0$ σ -strongly as $t \rightarrow 0^+$. Now note that $\Pi_t^\# = (I + \check{\omega}|_t \Lambda)^{-1} \check{\omega}|_t$. Since the range of $\check{\omega}|_t$ is invariant under $I + \check{\omega}|_t \Lambda$, the same holds for the inverse. Hence, the range of $\Pi_t^\#$ is contained in the range of $\check{\omega}|_t$ and the latter is contained in the range of $\check{\omega}$ which is finite dimensional. Therefore, $\|\Pi_t^\#(E(s, \infty))\| \rightarrow 0$ as $t \rightarrow 0^+$.

The maps $\phi_t(A) := \Pi_t^\#(E(s, \infty)AE(s, \infty))$ are completely positive for all $t > 0$ and thus satisfy

$$\|\phi_t\| = \|\phi_t(I)\| = \|\Pi_t^\#(E(s, \infty)IE(s, \infty))\| = \|\Pi_t^\#(E(s, \infty))\|,$$

so for every contraction $A \in \mathfrak{B}(\mathfrak{H})$, we have

$$(13) \quad \left\| \Pi_t^\#(E(s, \infty)AE(s, \infty)) \right\| \leq \|\Pi_t^\#(E(s, \infty))\|.$$

Putting together equations (12) and (13), we observe that if $t < s$, then

$$\begin{aligned} \|(I + \check{\omega}|_t \circ \Lambda)^{-1} \circ \check{\omega}|_s\| &= \sup_{\|A\| \leq 1, A \in \mathfrak{B}(\mathfrak{H})} \|(I + \check{\omega}|_t \circ \Lambda)^{-1} \check{\omega}|_s(A)\| \\ &= \sup_{\|A\| \leq 1, A \in \mathfrak{B}(\mathfrak{H})} \|(I + \check{\omega}|_t \circ \Lambda)^{-1} \check{\omega}|_t(E(s, \infty)AE(s, \infty))\| \\ &= \sup_{\|A\| \leq 1, A \in \mathfrak{B}(\mathfrak{H})} \|\Pi_t^\#(E(s, \infty)AE(s, \infty))\| \\ &= \|\Pi_t^\#(E(s, \infty))\| \rightarrow 0 \text{ as } t \rightarrow 0, \end{aligned}$$

establishing equation (11). Thus, for all $x \in \mathfrak{B}(\mathfrak{K})$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$,

$$(14) \quad \begin{aligned} \rho(L \circ \check{\omega}|_s(x)) &= \lim_{\mu} \rho\left([(I + \check{\omega}|_{t_\mu} \Lambda)^{-1} \check{\omega}|_{t_\mu} \Lambda](\check{\omega}|_s(x))\right) = \lim_{\mu} \rho\left([I - (I + \check{\omega}|_{t_\mu} \Lambda)^{-1}](\check{\omega}|_s(x))\right) \\ &= \rho(\check{\omega}|_s(x)) - \lim_{\mu} \rho\left((I + \check{\omega}|_{t_\mu} \Lambda)^{-1}(\check{\omega}|_s(x))\right) = \rho(\check{\omega}|_s(x)). \end{aligned}$$

Therefore, L fixes the range of $\check{\omega}|_s$ for every $s > 0$. Let $R \in \mathfrak{A}(\mathfrak{H})$, so $R = (I - \Lambda(I))^{\frac{1}{2}} B (I - \Lambda(I))^{\frac{1}{2}}$ for some $B \in \mathfrak{B}(\mathfrak{H})$. Let $\rho \in \mathfrak{B}(\mathfrak{K})_*$. By Proposition 2.9, we have $\lim_{s \rightarrow 0^+} \omega(\rho)(E(s, \infty)RE(s, \infty)) =$

$\omega(\rho)(R)$, hence $\lim_{s \rightarrow 0^+} \rho(\tilde{\omega}|_s(R)) = \rho(\tilde{\omega}(R))$. Since the range of $\tilde{\omega}|_t$ is contained in the range of $\tilde{\omega}$ for every $t > 0$ and the latter is finite dimensional we actually have that

$$(15) \quad \lim_{s \rightarrow 0^+} \|\tilde{\omega}|_s(R) - \tilde{\omega}(R)\| = 0.$$

Since $L \in \mathfrak{C}(\mathfrak{B}(\mathfrak{K}))$, it is norm-continuous, hence equations (14) and (15) imply

$$L(\tilde{\omega}(R)) = \lim_{s \rightarrow 0^+} L(\tilde{\omega}|_s(R)) = \lim_{s \rightarrow 0^+} \tilde{\omega}|_s(R) = \tilde{\omega}(R)$$

for all $R \in \mathfrak{B}(\mathfrak{H})$, thereby proving (ii).

Since L fixes the range of $\tilde{\omega}$ we have $\text{Range}(L) \supseteq \text{Range}(\tilde{\omega})$. From the algebraic fact that $\tilde{\omega}|_t \circ \Lambda$ commutes with $(I + \tilde{\omega}|_t \circ \Lambda)^{-1}$ for all $t > 0$, we have $x \in \mathfrak{B}(\mathfrak{K})$,

$$L(x) = \sigma\text{-weak-}\lim_{\mu} (I + \tilde{\omega}|_{t_\mu} \Lambda)^{-1} \tilde{\omega}|_{t_\mu} \Lambda(x) = \sigma\text{-weak-}\lim_{\mu} \tilde{\omega}|_{t_\mu} [\Lambda(I + \tilde{\omega}|_{t_\mu} \Lambda)^{-1}(x)].$$

Therefore, every element of the range of L is the σ -weak limit of elements in the range of $\tilde{\omega}$, which is a finite dimensional subspace of $\mathfrak{B}(\mathfrak{K})$ and is thus σ -weakly closed. Therefore, $\text{Range}(L) \subseteq \text{Range}(\tilde{\omega})$. We conclude $\text{Range}(L) = \text{Range}(\tilde{\omega})$, proving (iii), whereby property (ii) implies that L fixes its range, hence $L^2 = L$. Note $\|L\| \leq 1$ since $L \in \mathfrak{C}(\mathfrak{B}(\mathfrak{K}))$, so since L is idempotent we have $\|L\| = 0$ or $\|L\| = 1$. By assumption, $\tilde{\omega}$ is not the zero map, hence $\{0\} \subsetneq \text{Range}(\tilde{\omega}) = \text{Range}(L)$, so $\|L\| = 1$, proving (iv). \square

Let L be a boundary expectation for a q -weight ω over \mathfrak{K} , and let $\mathfrak{K}_L = L(\mathfrak{B}(\mathfrak{K}))$. Since $L : \mathfrak{B}(\mathfrak{K}) \rightarrow \mathfrak{B}(\mathfrak{K})$ is a completely positive and contractive idempotent, by the work of Choi-Effros [5] (see also section 6.1 of [6]), we recall that

$$(16) \quad L(L(T)S) = L(L(T)L(S)) = L(TL(S)), \quad \forall T, S \in \mathfrak{B}(\mathfrak{K}).$$

Furthermore, we have that \mathfrak{K}_L is a unital C^* -algebra under its norm and involution as a subspace of $\mathfrak{B}(\mathfrak{K})$ but with multiplication \otimes given by

$$(17) \quad x \otimes y = L(xy), \quad \forall x, y \in \mathfrak{K}_L.$$

If the range of L is σ -weakly closed (for example if it is finite-dimensional as in the previous theorem), then it is the dual of a Banach space hence \mathfrak{K}_L is a W^* -algebra. We also note that its unit is $L(I)$ (thus we remark that while \mathfrak{K}_L is unital, in general it does not share the unit with $\mathfrak{B}(\mathfrak{K})$).

Finally, we note that although a boundary expectation L need *not* be a conditional expectation in the traditional sense, it satisfies the following property by a direct application of (16):

$$L(xTy) = x \otimes L(T) \otimes y, \quad \forall T \in \mathfrak{B}(\mathfrak{K}), \forall x, y \in \mathfrak{K}_L.$$

The following lemma will be useful for the study of q -corners between q -weights with range rank one.

Lemma 3.3. *Let \mathfrak{H}_1 and \mathfrak{H}_2 be orthogonal complementary subspaces of \mathbb{C}^n . Suppose that $L : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a completely positive contractive idempotent map, which has block form*

$$L \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} L_{11}(A) & L_{12}(B) \\ L_{21}(C) & L_{22}(D) \end{pmatrix}$$

where $L_{ij} : \mathfrak{B}(\mathfrak{H}_j, \mathfrak{H}_i) \rightarrow \mathfrak{B}(\mathfrak{H}_j, \mathfrak{H}_i)$ for $i, j = 1, 2$. If $\dim(\text{Range } L_{11}) = \dim(\text{Range } L_{22}) = 1$, then either $L_{12} \equiv 0$ or $\dim(\text{Range } L_{12}) = 1$.

Proof. Suppose $L_{12} \not\equiv 0$, and let $B_0 \in \text{Range}(L_{12})$ be an element with $\|B_0\| = 1$. Notice that $B_0 = L_{12}(B_0)$.

We will show that the element of \mathfrak{K}_L given by

$$u = \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & L_{12}(B_0) \\ 0 & 0 \end{pmatrix} = L \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix}$$

is a partial isometry such that $u^* \otimes u + u \otimes u^* = L(I) = I_{\mathfrak{K}_L}$ and $u \otimes u^*$ is a minimal projection in the W^* -algebra \mathfrak{K}_L . Notice that $\|u\| = 1$ since the norm in \mathfrak{K}_L is the same as the norm as a subspace of $M_n(\mathbb{C})$. We note that

$$u \otimes u^* = L \left(\begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ B_0^* & 0 \end{pmatrix} \right) = L \begin{pmatrix} B_0 B_0^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{11}(B_0 B_0^*) & 0 \\ 0 & 0 \end{pmatrix}$$

$$u^* \otimes u = L \left(\begin{pmatrix} 0 & 0 \\ B_0^* & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix} \right) = L \begin{pmatrix} 0 & 0 \\ 0 & B_0^* B_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & L_{22}(B_0^* B_0) \end{pmatrix}$$

Since $\|u\| = 1$, we have that $\|L_{11}(B_0 B_0^*)\| = \|L_{22}(B_0^* B_0)\| = 1$. Let $T_1 = L_{11}(B_0 B_0^*)$ and $T_2 = L_{22}(B_0^* B_0)$, and note that these are positive operators since L is completely positive. Notice that since $\dim(\text{Range}(L_{11})) = \dim(\text{Range}(L_{22})) = 1$, and for each $i = 1, 2$, $T_i \in \text{Range}(L_{ii})$, hence it follows that there exists a linear functional ρ_i on $\mathfrak{B}(\mathfrak{H}_i)$ such that $L_{ii}(A) = \rho_i(A)T_i$ for all $A \in \mathfrak{B}(\mathfrak{H}_i)$. Since L is a completely positive contractive idempotent, so is L_{ii} , hence ρ_i is a state. Therefore, we have that

$$(u \otimes u^*) \otimes (u \otimes u^*) = L \left(\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \right) = L \begin{pmatrix} T_1^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \rho_1(T_1^2)T_1 & 0 \\ 0 & 0 \end{pmatrix} = \rho_1(T_1^2)(u \otimes u^*)$$

On the other hand, we have that $\|u \otimes u^*\| = 1$, hence by taking norms on both sides and using the C^* -norm identity, we obtain that $\rho_1(T_1^2) = 1$ and $u \otimes u^*$ is a projection. Hence u is a partial isometry in \mathfrak{K}_L .

Furthermore, observe that

$$\begin{aligned} I_{\mathfrak{K}_L} &= L(I) = \begin{pmatrix} L_{11}(I_{\mathfrak{H}_1}) & L_{12}(0) \\ L_{21}(0) & L_{22}(I_{\mathfrak{H}_2}) \end{pmatrix} = \begin{pmatrix} \rho_1(I_{\mathfrak{H}_1})T_1 & 0 \\ 0 & \rho_2(I_{\mathfrak{H}_2})T_2 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \\ &= u^* \otimes u + u \otimes u^*. \end{aligned}$$

We prove that $u \otimes u^*$ is minimal. Indeed, if q is a projection in \mathfrak{K}_L such that $q \leq u \otimes u^*$, then $(u \otimes u^*) \otimes q \otimes (u \otimes u^*) = q$. Now note that

$$\begin{aligned} (u \otimes u^*) \otimes q \otimes (u \otimes u^*) &= L((u \otimes u^*)q(u \otimes u^*)) = L \left(\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \right) \\ &= L \begin{pmatrix} T_1 q_{11} T_1 & 0 \\ 0 & 0 \end{pmatrix} = \rho_1(T_1 q_{11} T_1) u \otimes u^*. \end{aligned}$$

Since $q \leq u \otimes u^*$ is a projection, this implies that either $q = 0$ or $q = u \otimes u^*$.

Since u is a partial isometry such that $u^* \otimes u + u \otimes u^* = 1_{\mathfrak{K}_L}$ and $u \otimes u^*$ is a minimal projection, we obtain a system of matrix units $(e_{ij} : i, j = 1, 2)$ for \mathfrak{K}_L by assigning $e_{12} = u$ and following the relations

$$e_{ij} \otimes e_{k\ell} = \delta_{jk} e_{i\ell}, \quad e_{ij} = e_{ji}^*, \quad e_{11} + e_{22} = 1_{\mathfrak{K}_L}$$

for all $i, j = 1, 2$. Therefore, we have that for every $x \in \mathfrak{K}_L$, if we denote by $x_{ij} = e_{ii} x e_{jj}$, we have that $x = \sum_{i,j=1}^2 x_{ij}$. Note however that $x_{ij} e_{ji} \in e_{ii} \mathfrak{K}_L e_{ii}$. Since e_{11} is minimal, and e_{22} is Murray-von Neumann equivalent to e_{11} , we have that e_{22} is also minimal. Hence $e_{ii} \mathfrak{K}_L e_{ii} = \mathbb{C} e_{ii}$ for $i = 1, 2$. Therefore, there exists $\lambda_{ij} \in \mathbb{C}$ such that

$$x_{ij} e_{ji} = \lambda_{ij} e_{ii} \implies x = x_{ij} e_{ji} e_{ij} = \lambda_{ij} e_{ii} e_{ij} = \lambda_{ij} e_{ij}$$

Therefore, $x = \sum_{ij} \lambda_{ij} e_{ij}$ for some $\lambda_{ij} \in \mathbb{C}$ for $i = 1, 2$. In particular, for every $X \in M_n(\mathbb{C})$, $L_{12}(X) = \lambda_{12} e_{12}$ for some $\lambda_{12} \in \mathbb{C}$, hence $\text{Range}(L_{12}) = \text{span}(e_{12})$ and it is one-dimensional. \square

We will find in Theorem 6.5 that in the special case when $\mathfrak{K} = \mathbb{C}^2$, Theorem 3.2 and Lemma 3.3 can also be used to narrow down the possible range ranks of q -weights that are q -pure.

4. CP-SEMIGROUPS AND q -PURITY

Definition 4.1. We will say that a CP-flow α , or alternatively its q -weight map, is q -pure if its set of flow subordinates is totally ordered by subordination.

We remark that a unital CP-flow is q -pure if and only if its minimal flow dilation is also q -pure by Theorem 3.5 and Lemma 4.50 of [15]. We also note that a q -pure E_0 -semigroup must have index 0 or 1. However, since automorphism groups are not CP-flows, therefore a q -pure E_0 -semigroup cannot be of type I_0 .

Proposition 4.2. *Let α be a unital CP-flow over \mathfrak{K} which is q -pure, and let \mathfrak{S} be its set of flow subordinates. Then \mathfrak{S} is a complete totally ordered set which is order isomorphic (hence homeomorphic in the order topology) to a compact subset of the unit interval. Furthermore, the order topology on \mathfrak{S} can also be described by the uniform convergence in the strong operator topology on compact sets.*

Proof. Let α^d be the minimal flow dilation of α on the Hilbert space \mathfrak{H} . Then the set of subordinates of α is order isomorphic to the set \mathfrak{Q} of positive contractive flow cocycles of α^d . Notice that it is clear that if $T^j = (T_t^j)$ is an increasing net of positive contractive flow cocycles of α^d , then for each $t > 0$, the net T_t^j converges in the strong topology to an operator T_t , and $T = (T_t)$ must be a local positive contractive cocycle. The same argument applies for decreasing nets, hence \mathfrak{Q} is a complete totally ordered set. Now let $(e_n : n \in \mathbb{N})$ be an orthonormal basis for \mathfrak{H} , and define the map $\phi : \mathfrak{Q} \rightarrow [0, 1]$ by

$$\phi(T) = \sum_{n \in \mathbb{N}} 2^{-n} \int_0^1 \langle T_t e_n, e_n \rangle dt$$

It is clear that ϕ is a well-defined injective order-preserving map. Since \mathfrak{Q} is complete, it follows that the range of ϕ must be compact. See also Theorem 12, page 242 of [4].

Finally, note that the map ϕ is also continuous when \mathfrak{Q} is endowed with the convergence in the strong operator topology on compact sets. In this topology, which coincides with uniform σ -weak convergence on compact sets, the set \mathfrak{Q} is also compact, hence we obtain the desired homeomorphism. \square

We remark that in general we do not know whether \mathfrak{S} is homeomorphic to an *interval*. For that to hold, it suffices to show that for every two subordinates β and γ such that $\beta_t \leq \gamma_t$ for all t , there exists another subordinate σ such that $\beta_t \leq \sigma_t \leq \gamma_t$ for all $t > 0$ (see Theorem 14 of page 243 of [4]).

Of course the situation with E-subordinates of an E_0 -semigroup is somewhat simpler, in that they form a complete lattice (see Theorem 4.9 of [13]; see also Theorem 4 of [11]). We do not know if the analogous result holds for CP-flow subordinates of a unital CP-flow.

Concrete description of q -purity in the range rank one case. Let \mathfrak{K} be a separable Hilbert space and define $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$. If $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ is a q -weight map of range rank one (see Definition 2.12), then there exists a positive boundary weight $\mu \in \mathfrak{A}(\mathfrak{H})_*$ and a positive $T \in \mathfrak{B}(\mathfrak{K})$ such that

$$(18) \quad \omega(\rho)(A) = \rho(T)\mu(A)$$

for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$. However not every boundary weight map of the form (18) is a q -weight.

Theorem 4.3. *Suppose T is a positive operator in $\mathfrak{B}(\mathfrak{K})$ of norm one and $\mu \in \mathfrak{A}(\mathfrak{H})_*$ is positive and $\mu(I - \Lambda(T)) \leq 1$. Then the mapping $\omega(\rho)$ of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{A}(\mathfrak{H})_*$ given by*

$$\omega(\rho)(A) = \rho(T)\mu(A)$$

$\rho \in \mathfrak{B}(\mathfrak{K})_$ and for $A \in \mathfrak{A}(\mathfrak{H})$ is a q -weight. The q -weight map ω is unital if and only if $T = I$ and $\mu(I - \Lambda(T)) = 1$. Conversely every q -weight of range rank one is of the above form, and its generalized boundary representation $\pi^\#$ is given by*

$$\pi_t^\#(A) = \frac{\mu|_t(A)}{1 + \mu|_t(\Lambda(T))} T, \quad \forall t > 0, \forall A \in \mathfrak{B}(\mathfrak{H}).$$

Proof. Suppose ω is a boundary weight map of the form (18), where $\mu \in \mathfrak{A}(\mathfrak{H})_*$ is positive and $T \in \mathfrak{B}(\mathfrak{K})$ is positive with norm one. We observe that

$$\hat{\Lambda}\omega|_t(\rho)(A) = \rho(T) \cdot \mu|_t(\Lambda(A))$$

for all $t > 0$, $A \in \mathfrak{B}(\mathfrak{K})$, and $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Then we see that the inverse of the mapping

$$A \rightarrow A + \mu|_t(\Lambda(A)) \cdot T$$

is given by

$$A \rightarrow A - (1 + \mu|_t(\Lambda(T)))^{-1} \mu|_t(\Lambda(A)) \cdot T.$$

Then the generalized boundary representation is

$$\begin{aligned} \pi_t^\#(A) &= \mu|_t(A) \cdot T - (1 + \mu|_t(\Lambda(T)))^{-1} \mu|_t(T) \mu|_t(A) \cdot T \\ &= (1 + \mu|_t(\Lambda(T)))^{-1} \mu|_t(A) \cdot T. \end{aligned}$$

We see that the generalized boundary representation is completely positive for all $t > 0$. Now we need to check that the norm $\pi_t^\#(I)$ is not greater than one. We have

$$\pi_t^\#(I) = (1 + \mu|_t(\Lambda(T)))^{-1} \mu|_t(I) \cdot T$$

and since T is of norm one we have

$$\|\pi_t^\#(I)\| = (1 + \mu|_t(\Lambda(T)))^{-1} \mu|_t(I)$$

for all $t > 0$. In order that this norm not exceed one we must have

$$\mu|_t(I - \Lambda(T)) \leq 1$$

and since the above function of t is non increasing the above inequality holds for all $t > 0$ if and only if

$$\mu(I - \Lambda(T)) \leq 1.$$

Since $\Lambda(T) \leq \Lambda(I) = \Lambda$ it follows that this inequality implies $\mu(I - \Lambda) \leq 1$ so even if we did not assume $\mu \in \mathfrak{A}(\mathfrak{H})_*$ the conditions that ω be a contractive q -positive boundary weight map would force this on us. \square

Theorem 4.4. *Let \mathfrak{K} be a separable Hilbert space, define $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$ and let $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ be a non-zero q -weight map of range rank one. Then for all $\lambda \in [0, 1]$, the map $\lambda\omega$ is a q -weight map subordinate to ω . Furthermore, ω is q -pure if and only if every q -subordinate of ω has the form $\lambda\omega$ for some $\lambda \in [0, 1]$.*

Proof. Let $\lambda \in [0, 1]$ be given. Let us show that if ω is a q -weight of range rank one then $\lambda\omega$ is a q -subordinate of ω . Let $\mu \in \mathfrak{A}(\mathfrak{H})_*$ be positive and $T \in \mathfrak{B}(\mathfrak{K})$ positive with norm one be given by the previous theorem so that ω satisfies (18). Then by the previous theorem $\lambda\omega$ is a q -weight map of range rank one. Let $\pi^\#$ and $\phi^\#$ be the generalized boundary representations of ω and $\lambda\omega$, respectively. Then we have that for all $t > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$,

$$\begin{aligned} \pi_t^\#(A) &= (1 + \mu|_t(\Lambda(T)))^{-1} \mu|_t(A) \cdot T, \\ \phi_t^\#(A) &= (1 + \lambda\mu|_t(\Lambda(T)))^{-1} \lambda\mu|_t(A) \cdot T. \end{aligned}$$

Since for $b \geq 0$ the function $h(x) = x/(1 + bx)$ is an increasing function of x it follows that $\pi_t^\# \geq \phi_t^\#$ for all $t > 0$. Thus we have that $\lambda\omega \leq_q \omega$.

Now suppose that ω is non-zero and q -pure and η is a non-zero q -subordinate of ω . Let $\check{\omega}$ and $\check{\eta}$ be the dualized q -weight maps corresponding to η and ω , respectively.

Observe that if $\omega_1 \leq_q \omega_2$ are q -weights, then

$$(19) \quad \check{\omega}_1(I - \Lambda) \leq \check{\omega}_2(I - \Lambda).$$

Thus we have that that $0 \leq \check{\eta}(I - \Lambda) \leq \check{\omega}(I - \Lambda)$, and observe that $\check{\omega}(I - \Lambda) \neq 0$ and $\check{\eta}(I - \Lambda) \neq 0$ since ω and η are non-zero. Let

$$\lambda = \frac{\|\check{\eta}(I - \Lambda)\|}{\|\check{\omega}(I - \Lambda)\|}$$

and notice that $0 < \lambda \leq 1$. If $\lambda = 1$, then by Proposition 2.17, we have that $\eta = \omega$, as desired. So let us consider the case $0 < \lambda < 1$. Let $\varepsilon > 0$ be small enough so that $0 < \lambda - \varepsilon < \lambda + \varepsilon < 1$ and

$$(20) \quad (\lambda - \varepsilon) \|\check{\omega}(I - \Lambda)\| < \|\check{\eta}(I - \Lambda)\| < (\lambda + \varepsilon) \|\check{\omega}(I - \Lambda)\|.$$

Now observe that $(\lambda - \varepsilon)\omega \leq_q (\lambda + \varepsilon)\omega$. Furthermore we cannot have $\eta \leq_q (\lambda - \varepsilon)\omega$ or $\eta \geq_q (\lambda + \varepsilon)\omega$, because by (19) either inequality implies

$$\check{\eta}(I - \Lambda) \leq (\lambda - \varepsilon)\check{\omega}(I - \Lambda), \quad \text{or} \quad \check{\eta}(I - \Lambda) \geq (\lambda + \varepsilon)\check{\omega}(I - \Lambda),$$

in which case we have that

$$\|\check{\eta}(I - \Lambda)\| \leq (\lambda - \varepsilon)\|\check{\omega}(I - \Lambda)\|, \quad \text{or} \quad \|\check{\eta}(I - \Lambda)\| \geq (\lambda + \varepsilon)\|\check{\omega}(I - \Lambda)\|,$$

contradicting (20). However, by assumption ω is q -pure, hence the set of q -subordinates of ω is totally ordered. Therefore we must have

$$(21) \quad (\lambda - \varepsilon)\omega \leq_q \eta \leq_q (\lambda + \varepsilon)\omega.$$

Now by Proposition 2.17, this implies that for all $\varepsilon > 0$ small enough, positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and positive $T \in \mathfrak{A}(\mathfrak{H})$,

$$(\lambda - \varepsilon)\omega(\rho)(T) \leq \eta(\rho)(T) \leq (\lambda + \varepsilon)\omega(\rho)(T).$$

Thus we have that for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and positive $T \in \mathfrak{A}(\mathfrak{H})$, $\eta(\rho)(T) = \lambda\omega(\rho)(T)$ hence $\eta = \lambda\omega$. \square

We note that when $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ is a q -weight map but with range rank not equal to 1, then the above description of q -purity does not hold, for instance $\lambda\omega$ can fail to be a subordinate of ω for all $\lambda \in (0, 1)$ (see Section 4 of [14] or the beginning of Section 4 of [8]). Furthermore, if ν is a type II Powers weight such $A \mapsto \nu((I - \Lambda)^{1/2}A(I - \Lambda)^{1/2})$ is a pure normal state and $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is such that $\phi \circ (I \otimes \nu)$ defines a q -weight map, then ω is q -pure if and only if every non-zero q -subordinate η of ω has the form

$$\check{\eta} = \phi(I + t\phi)^{-1} \circ (I \otimes \nu)$$

for some $t \geq 0$ (see Definition 4.2 and Lemma 4.3 of [8]).

5. THE RANGE RANK ONE CASE

In this section we will study E_0 -semigroups arising from q -weights with range rank one and their subordinates. Before we begin, however, we need to analyze the subordination structure of boundary weights.

5.1. Boundary weights and their subordinates. Let \mathfrak{K} be a separable Hilbert space and let $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$, the space of \mathfrak{K} -valued Lebesgue measurable functions defined on $(0, \infty)$. We have found the following description of the boundary weights acting on $\mathfrak{A}(\mathfrak{H})$ to be useful. Let $q : (0, \infty) \rightarrow \mathbb{R}$ be given by $q(x) = 1 - e^{-x}$, and let $\mathfrak{H}_q = \mathfrak{K} \otimes L^2(0, \infty; q(x)dx)$ be the linear space of Lebesgue measurable \mathfrak{K} -valued functions which are square integrable with respect to the measure $(1 - e^{-x})dx$. Notice that the operator M_q of multiplication by $q(x)$ is bounded on \mathfrak{H}_q . We define a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{H}_q \times M_q\mathfrak{H}_q$ as follows: given $f \in \mathfrak{H}_q$ and $g \in M_q\mathfrak{H}_q$, then

$$\langle f, g \rangle = \int_0^\infty \overline{f(x)}g(x)dx.$$

Now observe that if $A \in \mathfrak{A}(\mathfrak{H})$, and $g \in \mathfrak{H}_q$, then we have that Ag is well-defined in a natural way and furthermore $Ag \in M_q\mathfrak{H}_q$. Now if $\omega \in \mathfrak{A}(\mathfrak{H})_*$ we have the functional $\rho(A) = \omega((I - \Lambda)^{1/2}A(I - \Lambda)^{1/2})$ for $A \in \mathfrak{B}(\mathfrak{H})$ is normal so there are two orthonormal sets of vectors $\{f_i, g_i, i = 1, 2, \dots\}$ and a sequence of positive real numbers $(\lambda_i)_{i=1}^r$ ($r = \infty$ is allowed) such that

$$\sum_{i=1}^r \lambda_i < \infty \quad \text{and} \quad \rho(A) = \sum_{i=1}^r \lambda_i \langle f_i, Ag_i \rangle$$

for $A \in \mathfrak{B}(\mathfrak{H})$. We can then think of the functions $h_i = (I - \Lambda)^{-\frac{1}{2}}f_i$ and $k_i = (I - \Lambda)^{-\frac{1}{2}}g_i$ as two sets of orthonormal vectors in \mathfrak{H}_q (with respect to the inner product of \mathfrak{H}_q and not the sesquilinear form) and in terms of these vectors we have

$$\omega(A) = \sum_{i=1}^r \lambda_i \langle h_i, Ak_i \rangle$$

for $A \in \mathfrak{A}(\mathfrak{K})$. Note if ω is positive then we can arrange it so $f_i = g_i$ or $h_i = k_i$.

We now define a useful ordering on positive boundary weights.

Definition 5.1. Suppose ω and η are positive boundary weights on $\mathfrak{A}(\mathfrak{H})$. We say ω q -dominates η or η is a q -subordinate of ω , written $\omega \geq_q \eta$ if for all $t > 0$ we have

$$\frac{\omega|_t(A)}{1 + \omega|_t(\Lambda)} \geq \frac{\eta|_t(A)}{1 + \eta|_t(\Lambda)}$$

for all positive $A \in \mathfrak{B}(\mathfrak{H})$.

Let $\omega \in \mathfrak{A}(\mathfrak{H})_*$ be a positive boundary weight. If $A \in \mathfrak{B}(\mathfrak{H})$ is positive and $\omega|_t(A) \rightarrow \infty$ as $t \rightarrow 0+$, we will write $\omega(A) = \infty$. This notation will be useful when dealing with unbounded boundary weights which are unbounded linear functionals (i.e. $\omega(I) = \infty$) or weights for which $\omega|_t(\Lambda(E)) \rightarrow \infty$ as $t \rightarrow 0+$ for a projection E .

One checks that if $\omega \geq_q \eta$ and $\eta \geq_q \mu$ then $\omega \geq_q \mu$. Also we see that the q -ordering is stronger than the normal ordering in that if $\omega \geq_q \eta$ then $\omega \geq \eta$ (i.e. $\omega(A) \geq \eta(A)$ for all positive $A \in \mathfrak{A}(\mathfrak{H})$). Note that if ω is a positive boundary weight then $\omega \geq_q \lambda\omega$ for all $\lambda \in [0, 1]$.

Definition 5.2 ([14]). We say a positive boundary weight ω is q -pure if $\omega \geq_q \eta \geq_q 0$ if and only if $\eta = \lambda\omega$ with $\lambda \in [0, 1]$.

We should remark that in the special case when $\mathfrak{K} = \mathbb{C}$, a q -weight map ω over \mathfrak{K} is q -pure (Definition 4.1) if and only if $\omega(1)$ is q -pure as a boundary weight.

Note the q -ordering is quite different from the normal ordering. For example the sum $\omega + \eta$ of two positive weights can be q -pure. In Theorem 3.9 of [14] this order relation was characterized in the case where \mathfrak{K} is one dimensional so $\mathfrak{H} = L^2(0, \infty)$. It turns out the same characterization applies to when \mathfrak{K} is any separable Hilbert space.

Theorem 5.3. Suppose ω and η are positive boundary weights on $\mathfrak{A}(\mathfrak{H})$. Suppose $\rho \in \mathfrak{A}(\mathfrak{H})_*$ is positive and $\omega \geq \rho$ and $\rho(I) < \infty$ (so $\rho \in \mathfrak{B}(\mathfrak{H})_*$) and $\eta = \lambda(1 + \rho(\Lambda))^{-1}(\omega - \rho)$ with $0 \leq \lambda \leq 1$. Then $\omega \geq_q \eta$. Conversely suppose $\omega \geq_q \eta$ and $\eta \neq 0$. Then there is a positive element $\rho \in \mathfrak{B}(\mathfrak{H})_*$ and a real number $\lambda \in (0, 1]$ so that $\eta = \lambda(1 + \rho(\Lambda))^{-1}(\omega - \rho)$. Furthermore, if $\omega(I) = \infty$ then ρ and λ are unique. It follows then that a positive boundary weight on $\mathfrak{A}(\mathfrak{H})$ is q -pure if and only if every rank one positive functional $\rho \in \mathfrak{B}(\mathfrak{H})_*$ subordinate to ω (so if $\omega \geq \rho$) is a multiple of ω . (Note in the case when $\omega(\Lambda) = \infty$ this means there are no bounded positive functionals subordinate to ρ)

Proof. Assume the first two sentences in the statement of the theorem are satisfied. Then we have

$$\begin{aligned} \frac{\eta|_t}{1 + \eta|_t(\Lambda)} &= \frac{\omega|_t - \rho|_t}{\lambda^{-1} + \lambda^{-1}\rho(\Lambda) + \omega|_t(\Lambda) - \rho|_t(\Lambda)} \\ &\leq \frac{\omega|_t - \rho|_t}{1 + \rho(\Lambda) + \omega|_t(\Lambda) - \rho|_t(\Lambda)} \leq \frac{\omega|_t - \rho|_t}{1 + \omega|_t(\Lambda)} \\ &\leq \frac{\omega|_t}{1 + \omega|_t(\Lambda)} \end{aligned}$$

Hence, we have

$$\frac{\omega|_t(A)}{1 + \omega|_t(\Lambda)} \geq \frac{\eta|_t(A)}{1 + \eta|_t(\Lambda)}$$

for all $t > 0$ so $\omega \geq_q \eta$.

Next assume ω and η are as stated in the first sentence of the statement of the theorem and $\omega \geq_q \eta \geq_q 0$. Let

$$h(t) = (1 + \eta|_t(\Lambda))/(1 + \omega|_t(\Lambda))$$

for $t > 0$. Since $\omega \geq_q \eta$ we have for $0 < t < s$

$$\begin{aligned} \frac{\eta|_t(\Lambda) - \eta|_s(\Lambda)}{1 + \eta|_t(\Lambda)} &= \frac{\eta|_t(E(t, s)\Lambda)}{1 + \eta|_t(\Lambda)} \\ &\leq \frac{\omega|_t(E(t, s)\Lambda)}{1 + \omega|_t(\Lambda)} = \frac{\omega|_t(\Lambda) - \omega|_s(\Lambda)}{1 + \omega|_t(\Lambda)}. \end{aligned}$$

Multiplying by the common denominator we have

$$\eta|_t(\Lambda) - \eta|_s(\Lambda) - \omega|_t(\Lambda)\eta|_s(\Lambda) \leq \omega|_t(\Lambda) - \omega|_s(\Lambda) - \eta|_t(\Lambda)\omega|_s(\Lambda).$$

Rearranging this inequality gives

$$\eta|_t(\Lambda) + \omega|_s(\Lambda) + \eta|_t(\Lambda)\omega|_s(\Lambda) \leq \omega|_t(\Lambda) + \eta|_s(\Lambda) + \omega|_t(\Lambda)\eta|_s(\Lambda),$$

and adding one gives

$$(1 + \eta|_t(\Lambda))(1 + \omega|_s(\Lambda)) \leq (1 + \omega|_t(\Lambda))(1 + \eta|_s(\Lambda)),$$

which yields $h(t) \leq h(s)$. Hence, h is non decreasing and since $h(t) \leq 1$ for all $t > 0$ the $h(t)$ has a limit as $t \rightarrow 0+$. We denote this limit by κ so $h(t) \rightarrow \kappa$ as $t \rightarrow 0+$. Since $\omega \geq_q \eta$ we have $h(t)\omega|_t \geq \eta|_t$ for all $t > 0$ so $\kappa\omega \geq \eta$. Now if $\kappa = 0$ then $\eta = 0$ and η is trivially a subordinate of ω . Since we are dealing with the case when $\eta \neq 0$ we have $\kappa > 0$. Let $\rho = \omega - \kappa^{-1}\eta$. Since $\kappa\omega \geq \eta$ we have ρ is positive. Note $\eta = \kappa(\omega - \rho)$ and $\omega \geq \rho$. Since h is non decreasing we have $h(t) \geq \kappa$ for all $t > 0$ and, hence,

$$\frac{1 + \kappa(\omega|_t(\Lambda) - \rho|_t(\Lambda))}{1 + \omega|_t(\Lambda)} \geq \kappa$$

for all $t > 0$. Hence, $\rho|_t(\Lambda) \leq \kappa^{-1} - 1$ for all $t > 0$ which yields $\rho(\Lambda) \leq \kappa^{-1} - 1$. Since $\rho \leq \omega$ we have $\rho(I - \Lambda) \leq \omega(I - \Lambda)$ and since ω is boundary weight we have $\omega(I - \Lambda) < \infty$ so $\rho(I) = \rho(I - \Lambda) + \rho(\Lambda) \leq \kappa^{-1} + \omega(I - \Lambda) - 1$. Hence, ρ is bounded so ρ is a positive element of $\mathfrak{B}(\mathfrak{H})_*$. Since $\kappa \leq (1 + \rho(\Lambda))^{-1}$ we have $\kappa = \lambda(1 + \rho(\Lambda))^{-1}$ with $\lambda \in (0, 1]$ and $\eta = \lambda(1 + \rho(\Lambda))^{-1}(\omega - \rho)$. Hence, η is of the form stated.

Finally, we show that if $\omega(I) = \infty$ then λ and ρ are unique. Suppose then that $\lambda, \lambda' \in (0, 1]$ and ρ and ρ' are positive elements of $\mathfrak{B}(\mathfrak{H})_*$ so that $\omega \geq \rho$ and $\omega \geq \rho'$ and $\eta = \lambda(1 + \rho(\Lambda))^{-1}(\omega - \rho)$ and $\eta = \lambda'(1 + \rho'(\Lambda))^{-1}(\omega - \rho')$. Then we have

$$(\lambda(1 + \rho(\Lambda))^{-1} - \lambda'(1 + \rho'(\Lambda))^{-1})\omega = \lambda(1 + \rho(\Lambda))^{-1}\rho - \lambda'(1 + \rho'(\Lambda))^{-1}\rho'$$

Note if the functional on the left is non-zero it is unbounded and the functional on the right is bounded so it follows both sides of the above equality are zero. Since $\omega \neq 0$ we have $\lambda(1 + \rho(\Lambda))^{-1} = \lambda'(1 + \rho'(\Lambda))^{-1}$ which when inserted in the right hand side of the above equality yields $\rho = \rho'$. Since $\rho(\Lambda) = \rho'(\Lambda)$ the fact that the right hand side is zero yields $\lambda = \lambda'$. Hence, in the case at hand λ and ρ are unique.

We prove the last statement of the theorem. From what we have proved we see that ω is q -pure if and only if every positive $\rho \in \mathfrak{B}(\mathfrak{H})_*$ with $\omega \geq \rho$ is a multiple of ω . Note every positive $\rho \in \mathfrak{B}(\mathfrak{H})_*$ can be written as a possibly infinite sum of positive multiples of orthogonal pure states of $\mathfrak{B}(\mathfrak{H})$ so for every positive $\rho \in \mathfrak{B}(\mathfrak{H})_*$ there is a pure state $\vartheta \in \mathfrak{B}(\mathfrak{H})_*$ (so $\vartheta(A) = (f, Af)$ for $A \in \mathfrak{B}(\mathfrak{H})$ where $f \in \mathfrak{H}$ is a unit vector) and a number $s > 0$ so that $\rho \geq s\vartheta$. It follows there are non zero positive $\rho \in \mathfrak{B}(\mathfrak{H})_*$ with $\omega \geq \rho$ if and only if there are non zero positive multiples of pure states $\vartheta \in \mathfrak{B}(\mathfrak{H})_*$ so that $\omega \geq s\vartheta \geq 0$. Then it follows that if $\omega(\Lambda) = \infty$ then ω is q -pure if and only if there for every pure state $\vartheta \in \mathfrak{B}(\mathfrak{H})_*$ so that $\omega \geq s\vartheta \geq 0$ then $s = 0$ and if $\omega(\Lambda) < \infty$ then ω is q -pure if and only if ω is pure in the ordinary sense of pure. \square

In light of Theorem 5.3, the proof of Theorem 3.10 of [14] can easily be adapted to give another characterization of q -pure boundary weights. Suppose

$$\omega(A) = \sum_{i=1}^r \lambda_i(f_i, Af_i)$$

for $A \in \mathfrak{A}(\mathfrak{H})$ where the $\{f_i\}$ are orthonormal in \mathfrak{H}_q and the λ_i are positive numbers whose sum is finite. Then ω is q -pure if and only if either $r = 1$ or for every set of complex numbers c_i for $0 < i < r + 1$ so that

$$0 < \sum_{i=1}^r |c_i|^2 < \infty \quad \text{the vector} \quad \sum_{i=1}^r c_i f_i \notin \mathfrak{H}.$$

In other words ω is q -pure if and only if it is pure in the ordinary sense or there is no linear combination of the f_i that lies in \mathfrak{H} . Now one easily sees the reason for the strange fact that the sum of two positive boundary weights can be q -pure. The boundary weight

$$\omega(A) = (f, Af) + (g, Ag)$$

for $A \in \mathfrak{A}(\mathfrak{H})$ and $f, g \in \mathfrak{H}_q$ fails to be q -pure if and only if $zf + g \in \mathfrak{H}$ for some complex number z . With a little thought it is easy to construct lots of examples of functions in \mathfrak{H}_q so that no linear combination of them is in \mathfrak{H} .

5.2. *q*-purity. We proceed to characterize the *q*-weight maps of range rank one which are *q*-pure, now that we are appropriately equipped with a concrete definition of *q*-purity in this case.

Theorem 5.4. *Let \mathfrak{K} be a separable Hilbert space. Suppose ω is a *q*-weight of range rank one over \mathfrak{K} so ω can be expressed in the form $\omega(\rho)(A) = \rho(T)\mu(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$ where T is a positive operator of norm one and μ is a positive element of $\mathfrak{A}(\mathfrak{H})_*$ and $\mu(I - \Lambda(T)) \leq 1$. Suppose $T_1 \in \mathfrak{B}(\mathfrak{K})$ is a positive norm one operator and $T_1 \leq T$. Let $\eta(\rho) = \lambda\rho(T_1)\mu(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ for $\lambda \geq 0$. Then η is a *q*-weight map which is *q*-subordinate to ω (i.e., $\omega \geq_q \eta$) if and only if $\lambda \leq (1 + \mu(\Lambda(T - T_1)))^{-1}$ where if $\mu(\Lambda(T - T_1)) = \infty$ then $\omega \geq_q \eta$ only for $\lambda = 0$.*

Proof. Assume ω and η are as stated in the theorem. Let $\pi^\#$ and $\phi^\#$ be the generalized boundary representations of ω and η , respectively. Then we have

$$\pi_t^\#(A) = (1 + \mu|_t(\Lambda(T)))^{-1} \mu|_t(A) \cdot T$$

and

$$\phi_t^\#(A) = (1 + \lambda\mu|_t(\Lambda(T_1)))^{-1} \lambda\mu|_t(A) \cdot T_1$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$. Since T_1 is positive and $\|T_1\| = 1$, there is a sequence of vector states ρ_n such that $\lim_{n \rightarrow \infty} \rho_n(T_1) = 1$, and since $T_1 \leq T \leq I$ we also have $\lim_{n \rightarrow \infty} \rho_n(T) = 1$. Therefore, to determine whether $\pi_t^\# \geq \phi_t^\#$ need apply ρ_o to $\pi_t^\#$ and $\phi_t^\#$. The $\omega \geq_q \eta$ if and only if

$$(1 + \mu|_t(\Lambda(T)))^{-1} \leq \lambda(1 + \lambda\mu|_t(\Lambda(T_1)))^{-1}$$

for all $t > 0$. Multiplying by the product of the denominators we find the above inequality is equivalent to

$$\lambda \leq (1 + \mu|_t(\Lambda(T - T_1)))^{-1}$$

and since $\mu|_t(\Lambda(T - T_1))$ increases as t decreases this inequality is valid for all t if and only if $\lambda \leq (1 + \mu(\Lambda(T - T_1)))^{-1}$. \square

Corollary 5.5. *Let \mathfrak{K} be a separable Hilbert space. Suppose ω is a *q*-weight of range rank one over \mathfrak{K} so ω can be expressed in the form $\omega(\rho)(A) = \rho(T)\mu(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$ where T is a positive operator of norm one and μ is a positive element of $\mathfrak{A}(\mathfrak{H})_*$ and $\mu(I - \Lambda(T)) \leq 1$. Suppose T is not a projection and let*

$$T = \int s dF(s)$$

be the spectral resolution of T . Since T is not a projection for some $s_o \in (0, 1)$ we have $F([s_o, 1]) \neq T$. Let $T_1 = F([s_o, 1])T$ and then let $\eta(\rho)(A) = \lambda\rho(T_1)\mu(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_$ and $A \in \mathfrak{A}(\mathfrak{H})$ where $\lambda \geq 0$. Then η is a *q*-weight with $\omega \geq_q \eta$ if and only if $0 \leq \lambda \leq 1/(1 + \mu(\Lambda(T - T_1)))$. There are always some positive λ satisfying this inequality since*

$$\mu(\Lambda(T - T_1)) \leq \kappa\mu(I - \Lambda(T)) \leq \kappa$$

*where $\kappa = s_o/(1 - s_o)$. It follows that if the boundary weight ω is *q*-pure then T is a projection.*

Proof. Except for the estimate of $\mu(\Lambda(T - T_1))$ the theorem follows immediately from Theorem 5.4. All that remains is proving the estimate for $\mu(\Lambda(T - T_1))$. Let $\kappa = s_o/(1 - s_o)$. We prove $\Lambda(T - T_1) \leq \kappa(I - \Lambda(T))$.

We note all the operators in the inequality we want to prove are multiplication operators. For example for $A \in \mathfrak{B}(\mathfrak{K})$ we have $(\Lambda(A)f) = e^{-x}Af(x)$ for $x \in [0, \infty)$ and $f \in \mathfrak{H}$. It follow that our inequality is valid if and only if it is valid for all x . Let $P = F([0, s_o]) = I - F([s_o, 1])$ so $T - T_1 \leq s_o P$ and $T_1 P = 0$ so $TP \leq s_o P$. Also we have

$$T = T(I - P) + TP \leq I - P + TP \leq I - P + s_o P$$

so

$$(1 - s_o)P \leq I - T$$

Now we have

$$\begin{aligned} \Lambda(T - T_1) &\leq s_o \Lambda(P) = \kappa(1 - s_o) \Lambda(P) = \kappa(1 - s_o) e^{-x} P \\ &\leq \kappa e^{-x} (I - T) = \kappa(e^{-x} - e^{-x} T) \leq \kappa(I - e^{-x} T) \\ &= \kappa(I - \Lambda(T)) \end{aligned}$$

for all $x > 0$. And so we have $\Lambda(T - T_1) \leq \kappa(I - \Lambda(T))$ which yields

$$\mu(\Lambda(T - T_1)) \leq \kappa\mu(I - \Lambda(T)) \leq \kappa.$$

If T is not a projection there is a T_1 satisfying the conditions of the theorem so that T_1 is not a multiple of T so ω is not q -pure. \square

Theorem 5.6. *Let \mathfrak{K} be a separable Hilbert space. Suppose ω is a q -weight of range rank one so ω can be expressed in the form $\omega(\rho)(A) = \rho(T)\mu(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$ where T is a positive operator of norm one and μ is a positive element of $\mathfrak{A}(\mathfrak{H})_*$ and $\mu(I - \Lambda(T)) \leq 1$. Suppose $\vartheta \in \mathfrak{B}(\mathfrak{H})_*$ is positive and $\mu \geq \vartheta \geq 0$. Let $\nu(A) = \lambda(\mu(A) - \vartheta(A))$ and let $\eta(\rho)(A) = \rho(T)\nu(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$. Then η is a q -weight subordinate to ω if and only if $0 \leq \lambda \leq 1/(1 + \vartheta(\Lambda(T)))$. Conversely, if we maintain the assumptions made on T and μ and we have $\eta(\rho) = \rho(T)\nu(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $\omega \geq_q \eta \geq_q 0$, then there is a positive $\vartheta \in \mathfrak{B}(\mathfrak{H})_*$ so that $\mu \geq \vartheta \geq 0$ and a number $\lambda \in [0, 1]$ so that $\nu = \lambda(1 + \vartheta(\Lambda(T)))^{-1}(\mu - \vartheta)$. Thus, if ω is q -pure then μ is q -pure.*

Proof. Assume the hypothesis and notation of the theorem is satisfied. Then if $\pi^\#$ and $\phi^\#$ are the generalized boundary representations of ω and η , respectively we have

$$\pi_t^\#(A) = (1 + \mu|_t(\Lambda(T)))^{-1} \mu|_t(A) \cdot T$$

and

$$\phi_t^\#(A) = (1 + \nu|_t(\Lambda(T)))^{-1} \nu|_t(A) \cdot T$$

for $t > 0$. Hence, $\omega \geq_q \eta$ if and only if

$$(22) \quad (1 + \nu|_t(\Lambda(T)))^{-1} \nu|_t \leq (1 + \mu|_t(\Lambda(T)))^{-1} \mu|_t$$

for all $t > 0$. If we replace $\Lambda(T)$ by $\Lambda(I)$ we have the above conditions says $\mu \geq_q \nu$. The analysis of the above order relations is almost identical to the analysis in Theorem 5.3 so rather than repeat that argument we leave it to the reader to check the conclusions are the same if one replaces Λ with $\Lambda(T)$. One important point to remember is that μ satisfies the condition $\mu(I - \Lambda(T)) \leq 1$. All we need is to assume $\mu(I - \Lambda(T)) < \infty$. We mention this because without this assumption the proof fails since $\mu(I - \Lambda(T))$ can be infinite for a boundary weight μ . (We have the condition $\mu(I - \Lambda(T)) \geq \mu(I - \Lambda)$ for positive boundary weights so a bound on $\mu(I - \Lambda)$ does not give us a bound on $\mu(I - \Lambda(T))$.) With this said the conclusion of the theorem follows. Note that for this new ordering given above a positive boundary weight satisfying $\mu(I - \Lambda(T)) < \infty$ is pure with respect to this new ordering if and only if μ is q -pure. So it follows that if ω is q -pure then μ is q -pure. \square

Now we have all the pieces to give necessary and sufficient conditions that a range rank one q -weight is q -pure.

We will make use of the following notation. If $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$, then we define the operator of $\mathfrak{B}(\mathfrak{H})$ given by

$$A|_t = E(t, \infty)AE(t, \infty).$$

We emphasize that in fact, for all $t > 0$, we have that $A|_t \in \mathfrak{A}(\mathfrak{H})$.

Theorem 5.7. *Let \mathfrak{K} be a separable Hilbert space. Suppose ω is a q -weight of range rank one over \mathfrak{K} so ω can be expressed in the form $\omega(\rho)(A) = \rho(T)\mu(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$ where T is a positive operator of norm one and μ is a positive element of $\mathfrak{A}(\mathfrak{H})_*$ and $\mu(I - \Lambda(T)) \leq 1$. Then ω is q -pure if and only if the following three conditions are met.*

- (i): T is a projection.
- (ii): μ is q -pure.
- (iii): If $\text{rank}(T) > 1$ and $e \in \mathfrak{B}(\mathfrak{K})$ is a rank one projection with $T \geq e$ then $\mu(\Lambda(e)) = \infty$.

Proof. Assume ω is of the form given in the statement of the theorem and suppose that ω is q -pure. Then from Corollary 5.5 we know that T is a projection hence condition (i) is satisfied. And from Theorem 5.6 we know that μ is q -pure, hence condition (ii) is satisfied. Suppose that $\text{rank}(T) > 1$ however condition (iii) is not satisfied. Then there exists a rank one projection $e \in \mathfrak{B}(\mathfrak{K})$ with $T \geq e$ and $\mu(\Lambda(e)) < \infty$. Let $T_1 = T - e$. Then $\Lambda(T - T_1) = \Lambda(e)$ and we have $\mu(\Lambda(T - T_1)) < \infty$. Thus by Theorem 5.4 there is a range rank one q -weight map η with $\omega \geq_q \eta$ such that η is not a multiple of ω .

Hence, ω is not q -pure. Since this contradicts our assumption, we conclude that if ω is q -pure, then it satisfies conditions (i), (ii) and (iii).

Now we assume ω is of the form given above and three conditions given above are satisfied and η is a q -weight map so that $\omega \geq_q \eta$. The proof will be complete when we show $\eta = \lambda\omega$ with $\lambda \in [0, 1]$. Let $\pi^\#$ and $\phi^\#$ be the generalized boundary representations of ω and η , respectively. Consider the mapping ψ of $\mathfrak{B}(\mathfrak{K})_*$ into itself given by $(\psi(\rho))(A) = \rho(TAT)$ for $A \in \mathfrak{B}(\mathfrak{K})$. Note ψ is completely positive and $\psi^2 = \psi$. We will show that $\eta(\rho) = \eta(\psi(\rho))$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Assume $\rho \in \mathfrak{B}(\mathfrak{K})$ and $A \in \mathfrak{A}(\mathfrak{H})$ and both ρ and A are positive. Since $\omega \geq_q \eta$ we have $\omega \geq \eta$ so $\omega(\rho)(A) = \rho(T)\mu(A) \geq \eta(\rho)(A)$. Hence, we have for positive $\rho \in \mathfrak{B}(\mathfrak{K})$ if $\rho(T) = 0$ then $\eta(\rho) = 0$. We write $\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$ with $\mathfrak{K}_1 = T\mathfrak{K}$ and $\mathfrak{K}_2 = (I - T)\mathfrak{K}$. Now each $X \in \mathfrak{B}(\mathfrak{K})$ can be uniquely expressed in matrix form

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

where X_{ij} maps \mathfrak{K}_i into \mathfrak{K}_j for $i, j = 1, 2$. Let $\tilde{\omega} : \mathfrak{A}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K})$ denote the dualized q -weight map for ω . For $A \in \mathfrak{A}(\mathfrak{H})$ we have that $\tilde{\omega}(A)$ can be expressed in matrix form as

$$\tilde{\omega}(A) = \begin{bmatrix} \mu(A)I & 0 \\ 0 & 0 \end{bmatrix}$$

Let $\check{\eta} : \mathfrak{A}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K})$ denote the dualized q -weight map for η . Since $\omega \geq \eta$, we have that $\tilde{\omega} \geq \check{\eta}$ hence

$$\tilde{\omega}(A) = \begin{bmatrix} \mu(A)I & 0 \\ 0 & 0 \end{bmatrix} \geq \check{\eta}(A) = \begin{bmatrix} \check{\eta}(A)_{11} & \check{\eta}(A)_{12} \\ \check{\eta}(A)_{21} & \check{\eta}(A)_{22} \end{bmatrix} \geq 0$$

for positive $A \in \mathfrak{A}(\mathfrak{H})$. This show us that $\check{\eta}(A)_{22} = 0$ for positive A and since $\check{\eta}(A)_{22} = 0$ the only way that the above matrix can be positive is for both $\check{\eta}(A)_{12} = 0$ and $\check{\eta}(A)_{21} = 0$. Hence, $\check{\eta}(A)_{12} = \check{\eta}(A)_{21} = \check{\eta}(A)_{22} = 0$ for all positive $A \in \mathfrak{A}(\mathfrak{H})$. Since every $A \in \mathfrak{A}(\mathfrak{H})$ is the complex linear combination of four positive elements of $\mathfrak{A}(\mathfrak{H})$ we have

$$\check{\eta}(A) = \begin{bmatrix} \check{\eta}(A)_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \forall A \in \mathfrak{A}(\mathfrak{H}).$$

Since the mapping $A \rightarrow TAT$ in matrix form sets all the A_{ij} equal to zero except A_{11} it follows that $\eta(\rho) = \eta(\psi(\rho))$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$.

Now let $\pi_t^\#$ and $\phi_t^\#$ be the generalized boundary representations of ω and η , respectively. Since $\omega \geq_q \eta$ we have

$$\pi_t^\#(\rho)(A) = \rho(T)(1 + \mu|_t(\Lambda(T)))^{-1}\mu|_t(A) \geq \eta((I + \hat{\Lambda}\eta|_t)^{-1}\rho)(A|_t) \geq 0$$

for positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$, positive $A \in \mathfrak{A}(\mathfrak{H})$ and $t > 0$. Now let us replace ρ in the above inequality by $\vartheta \in \mathfrak{B}(\mathfrak{K})_*$ given by $\vartheta(A) = \rho(A) + \eta(\rho)(\Lambda(A)|_t)$. Note for $\rho \geq 0$ we have $\vartheta \geq 0$. With this replacement we have

$$(\rho(T) + \eta(\rho)|_t(\Lambda(T)))(1 + \mu|_t(\Lambda(T)))^{-1}\mu(A) \geq \eta(\rho)|_t(A)$$

for positive $A \in \mathfrak{A}(\mathfrak{H})$, positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $t > 0$. Now assume further that ρ is a state and $\rho(T) = 1$. Then we have

$$(1 + \mu|_t(\Lambda(T)))^{-1}\mu|_t \geq (1 + \eta(\rho)|_t(\Lambda(T)))^{-1}\eta(\rho)|_t$$

for all $t > 0$. But this is exactly the situation we had with inequality (22) in the previous theorem with ν replaced by $\eta(\rho)$. Hence, we conclude $\eta(\rho)$ is of the form given in the previous theorem and since μ is q -pure we have $\eta(\rho) = \lambda\mu$ for some $\lambda \geq 0$. Hence, we have shown that $\eta(\rho) = \lambda\mu$ for each positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ so that $\rho(T) = \rho(I)$. Now for arbitrary positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we have $\eta(\rho) = \eta(\psi(\rho))$ and $\psi(\rho)$ is positive and $\psi(\rho)(T) = \psi(\rho)(I)$ so $\eta(\rho) = \lambda\mu$ for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Then by linearity we have $\eta(\rho) = \lambda\mu$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ where λ depends linearly on ρ and, therefore, there is an operator $C \in \mathfrak{B}(\mathfrak{K})$ so that $\eta(\rho) = \rho(C)\mu$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. From the positivity of η we see that $C \geq 0$ so we can write $C = \lambda T_1$ where $\lambda \geq 0$ and T_1 is a positive operator of norm one. If $\lambda = 0$ then $\eta = 0\omega = 0$ and the proof is complete so we assume $\lambda > 0$. Now we have shown that $\eta(\rho) = \eta(\psi(\rho))$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ from which it follows that $T_1 = TT_1T$ and since T is a projection and T_1 is of norm one we have $T \geq T_1 \geq 0$. If T_1 is a multiple of T the proof is complete so we assume T_1 is not a multiple of T . In this case $\text{rank}(T) \geq 2$ since in the rank one case T_1 must be a multiple of T . Now we can apply

Theorem 5.4 which tells us that $\lambda \leq (1 + \mu(\Lambda(T - T_1)))^{-1}$ so if $\lambda > 0$ we have $\mu(\Lambda(T - T_1)) < \infty$. Since T_1 is not a multiple of T we have $T - T_1$ is a positive non zero operator so there is a rank one hermitian $e \in \mathfrak{B}(\mathfrak{K})$ so that $\|T - T_1\|e \leq T - T_1$. Then we have $\|T - T_1\|\mu(\Lambda(e)) \leq \mu(\Lambda(T - T_1)) < \infty$ so condition (iii) is violated. Assuming T_1 was not a multiple of T has led to a contradiction so we conclude $T_1 = T$ and, therefore, $\eta = \lambda\omega$. \square

5.3. q -corners. Next we consider the problem of identifying the q -corners between range rank one q -weight maps. The first step is the observation that in the type Π_0 case, all q -corners between range rank one q -weight maps must have range rank one in the natural sense.

Theorem 5.8. *Let ω_1, ω_2 be q -weight maps of range rank one over finite-dimensional Hilbert spaces \mathfrak{K}_1 and \mathfrak{K}_2 respectively, whose associated normal spines are trivial. If γ is a non-zero q -corner from ω_1 to ω_2 , then it has range rank one, i.e. there exist $S \in \mathfrak{B}(\mathfrak{K}_2, \mathfrak{K}_1)$ and $\ell \in \mathfrak{A}(\mathfrak{H}_2, \mathfrak{H}_1)_*$ such that*

$$(23) \quad \gamma(\mu)(B) = \ell(B)\mu(S), \quad \forall B \in \mathfrak{A}(\mathfrak{H}_2, \mathfrak{H}_1), \forall \mu \in \mathfrak{B}(\mathfrak{K}_2, \mathfrak{K}_1)_*.$$

Proof. Suppose that γ is a q -corner from ω_1 to ω_2 and let ω be the q -weight map over $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ defined by

$$\omega = \begin{pmatrix} \omega_1 & \gamma \\ \gamma^* & \omega_2 \end{pmatrix}.$$

Notice that ω has finite range rank, since $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ is finite-dimensional. Let $\Pi_t^\#$ be the generalized boundary representation associated to ω . Notice that since the normal spines of the generalized boundary representations associated to ω_1 and ω_2 are zero, it follows that by positivity the same must hold for the normal spine of $\Pi_t^\#$. Thus by Theorem 3.2 there exists a boundary expectation for ω . Let us fix a boundary expectation L for ω obtained as in statement of Theorem 3.2, namely as a cluster point of $\Pi_t^\# \circ \Lambda$. Notice that in this case, for $j = 1, 2$, L_{jj} is a boundary expectation for ω_j . Since for each $j = 1, 2$, $\tilde{\omega}_j$ is range rank one and L_{jj} is a boundary expectation for ω_j , we have $\text{Range}(L_{jj}) = \text{Range}(\tilde{\omega}_j)$, it follows that $\dim(\text{Range}(L_{jj})) = 1$. Thus, by Lemma 3.3, we have that $\dim(\text{Range}(L_{12})) = 1$.

Since L is a boundary expectation for ω , we have that $\text{Range}(L) = \text{Range}(\tilde{\omega})$, therefore there exist $\ell \in \mathfrak{A}(\mathfrak{H}_2, \mathfrak{H}_1)_*$ and $S \in \mathfrak{B}(\mathfrak{K}_2, \mathfrak{K}_1)$ such that

$$\tilde{\omega} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \tilde{\omega}_1(A) & \ell(B)S \\ \ell^*(C)S^* & \tilde{\omega}_2(D) \end{pmatrix} = \begin{pmatrix} \rho_1(A)T_1 & \ell(B)S \\ \ell^*(C)S^* & \rho_2(D)T_2 \end{pmatrix}$$

Therefore, γ is given by (23), i.e. it is range rank one. \square

In the formal definition of a q -corner between two q -weights ω and η we assume ω and η are defined on different Hilbert spaces. In the following we find it is notationally more efficient to have the weights defined on the same Hilbert space. Naturally the weights need to live on orthogonal subspaces which we define as follows.

Definition 5.9. Suppose η_1 and η_2 are q -weight maps defined on $\mathfrak{A}(\mathfrak{H})$ with $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$ for \mathfrak{K} separable. We say η_1 and η_2 are *completely orthogonal* if there are orthogonal projections $E_1, E_2 \in \mathfrak{B}(\mathfrak{K})$ so that, if we denote $\psi_{ij}(\rho)(A) = \rho(E_i A E_j)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{B}(\mathfrak{K})$, then

$$\begin{aligned} \eta_1(\rho)(A) &= \eta_1(\psi_{11}(\rho))((E_1 \otimes I)A(E_1 \otimes I)), \\ \eta_2(\rho)(A) &= \eta_2(\psi_{22}(\rho))((E_2 \otimes I)A(E_2 \otimes I)) \end{aligned}$$

for $A \in \mathfrak{A}(\mathfrak{H})$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$.

A map $\gamma : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ is called an *internal q -corner* from η_1 and η_2 if for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$, $A \in \mathfrak{A}(\mathfrak{H})$,

$$\gamma(\rho)(A) = \gamma(\psi_{12}(\rho))((E_1 \otimes I)A(E_2 \otimes I))$$

and furthermore the map $\mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ given by

$$\rho \mapsto \eta_1(\rho) + \gamma(\rho) + \gamma^*(\rho) + \eta_2(\rho)$$

defines a q -weight map.

Remark 5.10. We will make use of the following observation. Let E_1, E_2 be orthogonal projection on \mathfrak{K} with $E_1 + E_2 = I$. Given a map $\phi : \mathfrak{A}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K})$, let $\phi_b : \mathfrak{A}(\mathfrak{H}) \rightarrow \mathfrak{B}(\mathfrak{K} \oplus \mathfrak{K})$ be the map given by

$$(24) \quad \phi_b(A) = \begin{bmatrix} E_1\phi(A)E & E_1\phi(A)E_2 \\ E_2\phi(A)E_1 & E_2\phi(A)E_2 \end{bmatrix}$$

Then ϕ is completely positive if and only if ϕ_b is completely positive. We will often refer to the map ϕ_b as the matricial notation for the map ϕ , and we will abbreviate it by identifying ϕ with ϕ_b when invoking formula (24).

We note that this definition is compatible with the earlier notion of q -corner.

Proposition 5.11. For $j = 1, 2$, Let \mathfrak{K}_j be a Hilbert space, let $\mathfrak{H}_j = \mathfrak{K}_j \otimes L^2(0, \infty)$ and let $\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$ and $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. For each $j = 1, 2$, let ω_j be a q -weight map over \mathfrak{K}_j , let E_j be the canonical projection from \mathfrak{K} onto \mathfrak{K}_j , and let η_j be the q -weight map over \mathfrak{K} given by

$$\eta_j(\rho)(A) = \omega_j(\rho_{jj})(A_{jj}), \quad \forall \rho \in \mathfrak{B}(\mathfrak{K})_*, \forall A \in \mathfrak{A}(\mathfrak{H}).$$

Then η_1 and η_2 are completely orthogonal, and moreover there exists a q -corner σ from ω_1 to ω_2 if and only if there exists an internal q -corner γ from η_1 to η_2 , and we have that

$$\gamma(\rho)(A) = \sigma(\rho_{12})(A_{12}), \quad \forall \rho \in \mathfrak{B}(\mathfrak{K})_*, A \in \mathfrak{A}(\mathfrak{H}).$$

Proof. Notice that for all $i, j = 1, 2$,

$$(25) \quad \rho_{ij} = [\psi_{ij}(\rho)]_{ij}, \quad A_{ij} = [(E_i \otimes I)A(E_j \otimes I)]_{ij},$$

therefore

$$\eta_j(\rho)(A) = \omega_j(\rho_{jj})(A_{jj}) = \eta_j(\psi_{jj}(\rho))((E_j \otimes I)A(E_j \otimes I))$$

hence η_1 and η_2 are completely orthogonal.

Suppose that σ is a q -corner from ω_1 to ω_2 . Let

$$\gamma(\rho)(A) = \sigma(\rho_{12})(A_{12}), \quad \forall \rho \in \mathfrak{B}(\mathfrak{K})_*, \forall A \in \mathfrak{A}(\mathfrak{H}).$$

Notice that by (25), we have that γ satisfies the equation

$$\gamma(\rho)(A) = \gamma(\psi_{12}(\rho))((E_1 \otimes I)A(E_2 \otimes I)), \quad \forall \rho \in \mathfrak{B}(\mathfrak{K})_*, \forall A \in \mathfrak{A}(\mathfrak{H}).$$

Now note that if we define $\eta : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ by $\eta(\rho) = \eta_1(\rho) + \gamma(\rho) + \gamma^*(\rho) + \eta_2(\rho)$, for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$, then

$$\eta(\rho)(A) = \omega_1(\rho_{11})(A_{11}) + \sigma(\rho_{12})(A_{12}) + \sigma^*(\rho_{21})(A_{21}) + \omega_2(\rho_{22})(A_{22})$$

for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$. Therefore, it follows immediately from the definition of q -corner, that η is a q -weight map over \mathfrak{K} .

Conversely, suppose that γ is an internal q -corner from η_1 to η_2 . Let $\sigma : \mathfrak{B}(\mathfrak{K}_2, \mathfrak{K}_1)_* \rightarrow \mathfrak{A}(\mathfrak{H}_2, \mathfrak{H}_1)_*$ be the map given by

$$\sigma(\mu_{12})(A_{12}) = \gamma(\mu)(A)$$

for all $\mu \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$. We show that σ is well-defined. First note that for every $\ell \in \mathfrak{B}(\mathfrak{K}_2, \mathfrak{K}_1)_*$ there exists $\mu \in \mathfrak{B}(\mathfrak{K})_*$ such that $\ell = \mu_{12}$ and for every $X \in \mathfrak{A}(\mathfrak{H}_2, \mathfrak{H}_1)_*$ there exists $A \in \mathfrak{A}(\mathfrak{H})$ such that $X = A_{12}$. Furthermore, since γ is an internal q -corner, for any other representatives μ' and A' such that $\ell = \mu'_{12}$ and $X = A'_{12}$, we have that

$$\psi_{12}(\mu) = \psi_{12}(\mu'), \quad (E_1 \otimes I)A(E_2 \otimes I) = (E_1 \otimes I)A'(E_2 \otimes I)$$

hence

$$\gamma(\mu')(A') = \gamma(\mu)(A),$$

thus σ is well-defined.

Now note that if we define once again $\eta : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ by $\eta(\rho) = \eta_1(\rho) + \gamma(\rho) + \gamma^*(\rho) + \eta_2(\rho)$, for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$, then we obtain once more

$$\eta(\rho)(A) = \omega_1(\rho_{11})(A_{11}) + \sigma(\rho_{12})(A_{12}) + \sigma^*(\rho_{21})(A_{21}) + \omega_2(\rho_{22})(A_{22})$$

or

$$\eta(\rho) = \begin{bmatrix} \omega_1(\rho_{11}) & \sigma(\rho_{12}) \\ \sigma^*(\rho_{21}) & \omega_2(\rho_{22}) \end{bmatrix}.$$

Therefore, if η is a q -weight, then σ is a q -corner from ω_1 to ω_2 . \square

The following lemma will be a useful tool in the remainder.

Lemma 5.12. *Let \mathfrak{K} be a Hilbert space, let $\sigma : \mathfrak{B}(\mathfrak{K}) \rightarrow M_2(\mathbb{C})$ be a $*$ -preserving linear map such that $\sigma_{12} \neq 0$, let $P_1, P_2 \in \mathfrak{B}(\mathfrak{K})$ be two non-zero orthogonal projections and let $Q \in \mathfrak{B}(\mathfrak{K})$ with norm one. Define $\phi : \mathfrak{B}(\mathfrak{K}) \rightarrow \mathfrak{B}(\mathfrak{K} \oplus \mathfrak{K})$ given by*

$$\phi(A) = \begin{bmatrix} \sigma_{11}(A)P_1 & \sigma_{12}(A)Q \\ \sigma_{21}(A)Q^* & \sigma_{22}(A)P_2 \end{bmatrix}$$

Then ϕ is (completely) positive if and only if $P_1QP_2 = Q$, $QQ^ \leq P_1$, $Q^*Q \leq P_2$, and σ is (completely) positive.*

Proof. Suppose that ϕ is positive. Since $\sigma \neq 0$, there exists $A_0 \geq 0$ such that $c_{12} = \sigma_{12}(A) \neq 0$. Notice that by positivity of ϕ and the fact that P_j is a nonzero projection, we have that $c_{jj} = \sigma_{jj}(A) > 0$ for $j = 1, 2$. By positivity of ϕ , we must have that for all $x, y \in \mathfrak{B}(\mathfrak{K})$,

$$|c_{12}|^2 |(Qx, y)|^2 \leq c_{11}c_{22}(P_1y, y)(P_2x, x)$$

Thus we have that $\ker P_2 \subseteq \ker Q$ and $\ker P_1 \subseteq \ker Q^*$, hence $P_2Q^* = Q^*$ and $P_1Q = P_1$, or $P_1QP_2 = Q$. Furthermore,

$$QQ^* = P_1QQ^*P_1 \leq P_1\|Q^*\|^2 = P_1$$

and similarly $Q^*Q \leq P_2$. Now let x_n be a sequence of unit vectors in $P_2\mathfrak{K}$ such that $\|Qx_n\| \rightarrow 1$, and let $y_n = Qx_n$ (without loss of generality they are nonzero vectors). Let U_n be the rank one partial isometry such that $U_nx_n = y_n/\|y_n\|$ and U_n is zero on $\{x_n\}^\perp$. If $G_n = U_n \oplus U_n^*$, then

$$\psi_n(A) = G_n^*\phi(A)G_n = \begin{bmatrix} \sigma_{11}(A)U_n^*U_n & \sigma_{12}(A)\|y_n\|U_n \\ \sigma_{21}(A)\|y_n\|U_n & \sigma_{22}(A)U_nU_n^* \end{bmatrix}$$

Since P_1 and P_2 are orthogonal, it is clear that the range of ψ_n is isomorphic to $M_2(\mathbb{C})$ and the map

$$A \mapsto \begin{bmatrix} \sigma_{11}(A) & \sigma_{12}(A)\|y_n\| \\ \sigma_{21}(A)\|y_n\| & \sigma_{22}(A) \end{bmatrix}$$

is (completely) positive if ϕ is (completely) positive. By taking limits, we obtain that σ is (completely) positive.

Conversely suppose that $P_1QP_2 = Q$, $QQ^* \leq P_1$, $Q^*Q \leq P_2$, and σ is (completely) positive. Then observe that

$$\phi(A) = \begin{bmatrix} \sigma_{11}(A)P_1 & \sigma_{12}(A)Q \\ \sigma_{21}(A)Q^* & \sigma_{22}(A)P_2 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \sigma_{11}(A)I & \sigma_{12}(A)Q \\ \sigma_{21}(A)Q^* & \sigma_{22}(A)P_2 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

But

$$\begin{bmatrix} \sigma_{11}(A)I & \sigma_{12}(A)Q \\ \sigma_{21}(A)Q^* & \sigma_{22}(A)P_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11}(A)I & \sigma_{12}(A)Q \\ \sigma_{21}(A)Q^* & \sigma_{22}(A)Q^*Q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{22}(A)(P_2 - Q^*Q) \end{bmatrix}$$

and

$$\begin{bmatrix} \sigma_{11}(A)I & \sigma_{12}(A)Q \\ \sigma_{21}(A)Q^* & \sigma_{22}(A)Q^*Q \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Q^* \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11}(A)I & \sigma_{12}(A)I \\ \sigma_{21}(A)I & \sigma_{22}(A)I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & Q \end{bmatrix}.$$

□

Lemma 5.13. *Let \mathfrak{K} be a finite-dimensional Hilbert space. Suppose ω and η are completely orthogonal q -pure range rank one q -weight maps over \mathfrak{K} so $\omega(\rho) = \rho(T_1)\mu$ and $\eta(\rho) = \rho(T_2)\nu$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and μ and ν are q -weights so that $\mu(I - \Lambda(T_1)) \leq 1$ and $\nu(I - \Lambda(T_2)) \leq 1$. Suppose that γ is a non-zero range rank one internal corner from ω to η . Then there exist $\tau \in \mathfrak{A}(\mathfrak{H})_*$ and $Q \in \mathfrak{B}(\mathfrak{K})$ satisfying*

$$(26) \quad \|Q\| = 1, \quad QQ^* \leq T_1 \quad \text{and} \quad Q^*Q \leq T_2$$

such that $\gamma(\rho) = \rho(Q)\tau$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_$. For $t > 0$ let*

$$h(t) = \frac{(1 + \mu|_t(\Lambda(T_1)))^{1/2} (1 + \nu|_t(\Lambda(T_2)))^{1/2}}{1 + \tau|_t(\Lambda(Q))}$$

Then $h(t)$ has a limit as $t \rightarrow 0+$, and furthermore $|h(t)|$ is a non-increasing function of t which is bounded above. If $\kappa = \lim_{t \rightarrow 0+} |h(t)|$, then $1 \leq |h(t)| \leq \kappa$ for all $t > 0$ and $\kappa\gamma$ is a ordinary internal corner from ω to η .

Conversely, given $Q \in \mathfrak{B}(\mathfrak{K})$ satisfying (26), let $\gamma : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ be given by $\gamma(\rho) = \rho(Q)\tau$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$, and let $h(t)$ be defined as above for $t > 0$. If there exists $\kappa \geq 1$ such that $|h(t)| \leq \kappa$ for all $t > 0$ and $\kappa\gamma$ is an ordinary internal corner from ω to η , then γ is an internal q -corner from ω to η . If ω and η are unbounded then the internal corner $\kappa\gamma$ from ω to η is trivially maximal in that $\lambda\kappa\gamma$ is not an internal corner from ω to η for $\lambda > 1$.

Proof. Suppose that γ is a non-zero range rank one internal q -corner from ω to η , so that the map $\Theta = \omega + \gamma + \gamma^* + \eta$ is a q -weight map from $\mathfrak{B}(\mathfrak{K})_*$ to $\mathfrak{A}(\mathfrak{H})_*$. We write Θ in matrix form

$$\Theta = \begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix}.$$

Since ω and η are range rank one, there exist T_1 and T_2 non-zero positive operators so that $\omega(\rho) = \rho(T_1)\mu$ and $\eta(\rho) = \rho(T_2)\nu$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and μ and ν are q -weights so that $\mu(I - \Lambda(T_1)) \leq 1$ and $\nu(I - \Lambda(T_2)) \leq 1$. Since they are q -pure, by Theorem 5.7 we have that T_1 and T_2 are projections. Furthermore, notice that if ω and η are completely orthogonal with respect to orthogonal projections E_1 and E_2 , then we automatically have that $T_j \leq E_j$ for $j = 1, 2$. Recall that without loss of generality, we may assume that $E_1 + E_2 = I$.

Suppose that γ has range rank one, i.e. there exists $Q \in \mathfrak{B}(\mathfrak{K})$ with norm one and $\tau \in \mathfrak{A}(\mathfrak{H})_*$ such that $\gamma(\rho) = \rho(Q)\tau$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Since it is an internal corner, we have that $E_1 Q E_2 = Q$. Thus, by Remark 5.10, $\check{\Theta}$ is completely positive if and only if the map

$$A \mapsto \begin{bmatrix} E_1 \check{\Theta}(A) E_1 & E_1 \check{\Theta}(A) E_2 \\ E_2 \check{\Theta}(A) E_1 & E_2 \check{\Theta}(A) E_2 \end{bmatrix} = \begin{bmatrix} \mu(A) T_1 & \tau(A) Q \\ \tau^*(A) Q^* & \nu(A) T_2 \end{bmatrix}$$

from $\mathfrak{A}(\mathfrak{H})$ to $\mathfrak{B}(\mathfrak{K} \oplus \mathfrak{K})$ is completely positive. It follows from Lemma 5.12 that Q satisfies (26). Thus we see any candidate for a range rank one q -corner must be of the form given in the statement of the lemma.

Calculating the generalized boundary representation of Θ we find

$$\Pi_t^\#(A) = \begin{bmatrix} (1 + \mu|_t(\Lambda(T_1)))^{-1} \mu|_t(A) T_1 & (1 + \tau|_t(\Lambda(Q)))^{-1} \tau|_t(A) Q \\ (1 + \tau^*|_t(\Lambda(Q^*)))^{-1} \tau^*|_t(A) Q^* & (1 + \nu|_t(\Lambda(T_2)))^{-1} \nu|_t(A) T_2 \end{bmatrix}.$$

To simplify notation we make the following definitions

$$\begin{aligned} a(t) &= 1 + \mu|_t(\Lambda(T_1)), \\ b(t) &= 1 + \nu|_t(\Lambda(T_2)), \\ c(t) &= 1 + \tau|_t(\Lambda(Q)). \end{aligned}$$

$$h(t) = \frac{\sqrt{a(t)b(t)}}{c(t)}.$$

By Lemma 5.12, $\Pi_t^\#$ is completely positive if and only if

$$A \mapsto \begin{bmatrix} (1 + \mu|_t(\Lambda(T_1)))^{-1} \mu|_t(A) & (1 + \tau|_t(\Lambda(Q)))^{-1} \tau|_t(A) \\ (1 + \tau^*|_t(\Lambda(Q^*)))^{-1} \tau^*|_t(A) & (1 + \nu|_t(\Lambda(T_2)))^{-1} \nu|_t(A) \end{bmatrix}$$

is completely positive. Since taking the Schur product of this mapping with a matrix

$$\begin{bmatrix} |x|^2 & xy \\ \overline{xy} & |y|^2 \end{bmatrix}$$

where $xy \neq 0$ preserves completely positivity, it follows that $\Pi_t^\#$ is completely positive if and only if

$$\psi_t(A) = \begin{bmatrix} \mu|_t(A) & |h(t)| \tau|_t(A) \\ |h(t)| \tau^*|_t(A) & \nu|_t(A) \end{bmatrix}$$

is completely positive for $t > 0$. Note in the above expression we can replace $|h(t)|$ by $h(t)$ since multiplying the upper off diagonal entry by z and the lower off diagonal entry by \overline{z} where $|z| = 1$ preserves complete positivity.

Returning to our proof we see that if γ is an internal q -corner then ψ_t is completely positive for every $t > 0$. We will now show that $|h(t)|$ is a non-increasing function of t and has a finite limit $\kappa \geq 1$ as $t \rightarrow 0+$. First we show $h(t)$ is bounded. To accomplish this we need to find a positive element

$A \in \mathfrak{A}(\mathfrak{H})$ so that $\mu(A) > 0$, $\nu(A) > 0$ and $\tau(A) \neq 0$. Since $\tau \neq 0$ we have $\tau|_{t_o} \neq 0$ for some $t_o > 0$. Hence there is an operator $B \in \mathfrak{B}(\mathfrak{H})$ with $\|B\| \leq 1$ so that $\tau(E(t, \infty)BE(t, \infty)) \neq 0$. Since μ and ν are not zero there is a $t_1 > 0$ so that $\mu|_{t_1}(I) > 0$ and $\nu|_{t_1}(I) > 0$. Let $s = \min(t_o, t_1)$ and let $A = 3E(s, \infty) + E(t_o, \infty)(zB + \bar{z}B^*)E(t_o, \infty)$ where $z \in \mathbb{C}$ and $|z| \leq 1$. Then we have $A \in \mathfrak{A}(\mathfrak{H})$ and $\mu(A) > 0$, $\nu(A) > 0$ and

$$\tau(A) = 3\tau(E(s, \infty)) + 2\operatorname{Re}(z\tau|_{t_o}(B)).$$

Since $\tau|_{t_o}(B) \neq 0$ we can arrange it so $\tau(A) \neq 0$ with an appropriate choice of z with $|z| \leq 1$. Since $A \geq 0$ we have $\psi_t(A) \geq 0$ for all $t > 0$ and for $0 < t \leq s$ we have $\psi_t(A)$ is constant. Since the determinant of ψ_t is positive we have

$$|h(t)\tau(A)|^2 \leq \mu(A)\nu(A)$$

for $t \in (0, s]$ and since $\tau(A) \neq 0$ we have $h(t)$ is bounded for $t \in (0, s]$ and h is clearly bounded for $t > s$ so h is bounded.

Let $A = T_1 + Q + Q^* + T_2$, which is a positive operator. For $0 < t < s$, we define $\Lambda_t^s(A) = E(t, s)\Lambda(A)E(t, s)$ and note that

$$\mu(\Lambda_t^s(A)) = \mu|_t(\Lambda(A)) - \mu|_s(\Lambda(A)).$$

The same applies to ν and τ . Then we have for $0 < t < s$ the matrix

$$\psi_t(\Lambda_t^s(A)) = \begin{bmatrix} \mu|_t(\Lambda_t^s(T_1)) & |h(t)|\tau|_t(\Lambda_t^s(Q)) \\ |h(t)|\tau^*|_t(\Lambda_t^s(Q^*)) & \nu|_t(\Lambda_t^s(T_2)) \end{bmatrix} = \begin{bmatrix} a(t) - a(s) & |h(t)|(c(t) - c(s)) \\ |h(t)|(\bar{c}(t) - \bar{c}(s)) & b(t) - b(s) \end{bmatrix}$$

is a positive matrix. Since the determinant is positive we have

$$|h(t)|^2|c(t) - c(s)|^2 \leq (a(t) - a(s))(b(t) - b(s))$$

for $0 < t < s$. Recalling the definition of $h(t)$ we have

$$(27) \quad |1 - c(s)/c(t)|^2 \leq (1 - a(s)/a(t))(1 - b(s)/b(t)).$$

$$\text{Now for } x, y \in [0, 1] \text{ we have } x - 2\sqrt{xy} + y = (\sqrt{x} - \sqrt{y})^2 \geq 0$$

which yields

$$(1 - \sqrt{xy})^2 = 1 - 2\sqrt{xy} + xy \geq 1 - x - y + xy = (1 - x)(1 - y).$$

Using this inequality with $x = a(s)/a(t)$ and $y = b(s)/b(t)$ in inequality (27) we find

$$(28) \quad |1 - c(s)/c(t)|^2 \leq (1 - \sqrt{\frac{a(s)b(s)}{a(t)b(t)}})^2$$

Note that $a(t) \geq a(s)$ and $b(t) \geq b(s)$ so we have

$$|1 - c(s)/c(t)| \leq (1 - \sqrt{\frac{a(s)b(s)}{a(t)b(t)}})$$

Since $1 - |z| \leq |1 - z|$ for $z \in \mathbb{C}$ we have

$$-\frac{|c(s)|}{|c(t)|} \leq -\frac{a(s)b(s)}{a(t)b(t)}.$$

Recalling the definition of h this inequality states $|h(t)| \geq |h(s)|$ and we have $|h(t)|$ is non increasing and since $|h(t)|$ is bounded above we have $|h(t)|$ converges to its least upper bound κ as $t \rightarrow 0+$. Hence, we have shown that $|h(t)| \rightarrow \kappa$ as $t \rightarrow 0+$ and $|h(t)| \leq \kappa$ for all $t > 0$. Since $h(t) \rightarrow 1$ as $t \rightarrow \infty$ we have $\kappa \geq 1$.

Now we show the function $h(t)$ has a limit as $t \rightarrow 0+$. Since the absolute value of h has a limit and that limit is not zero to show h has a limit it is enough to show the reciprocal has a limit as $t \rightarrow 0+$. Let $w(t) = (a(t)b(t))^{1/2}$ for $t > 0$. Multiplying inequality (28) by $|c(t)|^2$ we find

$$|c(t) - c(s)|^2 \leq |c(t)|^2(1 - w(s)/w(t))^2$$

and expanding both sides and canceling terms we find

$$-2\operatorname{Re}(c(t)c(s)) + |c(s)|^2 \leq |c(t)|^2 w(s)/w(t)(w(s)/w(t) - 2)$$

and dividing both sides by $w(t)w(s)$ we find

$$\frac{-2\operatorname{Re}(c(t)c(s))}{w(t)w(s)} + \frac{|c(s)|^2}{w(t)w(s)} \leq \frac{w(s)|c(t)|^2}{w(t)^3} - \frac{2|c(t)|^2}{w(t)^2}$$

Now

$$|1/h(t) - 1/h(s)|^2 = \frac{|c(t)|^2}{w(t)^2} + \frac{|c(s)|^2}{w(s)^2} - \frac{2\operatorname{Re}(c(t)c(s))}{w(t)w(s)}$$

and combining this with the above inequality we find

$$\frac{w(s)|c(t)|^2}{w(t)^3} - \frac{|c(t)|^2}{w(t)^2} + \frac{|c(s)|^2}{w(s)^2} - \frac{|c(s)|^2}{w(t)w(s)} \geq |1/h(t) - 1/h(s)|^2$$

And in recalling that $c(t)/w(t) = 1/h(t)$ this inequality becomes

$$(1 - w(s)/w(t))(|1/h(s)|^2 - |1/h(t)|^2) \geq |1/h(t) - 1/h(s)|^2$$

Since $w(s) \leq w(t)$ and $|h(t)| \rightarrow \kappa \geq 1$ as $t \rightarrow 0+$ it follows that the left hand side of the above inequality tends to zero as s and t approach zero so it follows that $1/h(t) - 1/h(s)$ tends to zero as s and t approach zero and since $1 \leq |h(t)| \leq \kappa$ it follows that $h(t) \rightarrow z\kappa$ as $t \rightarrow 0+$ where $|z| = 1$.

We have the map

$$\psi_t(A) = \begin{bmatrix} \mu|_t(A) & |h(t)|\tau|_t(A) \\ |h(t)|\tau^*|_t(A) & \nu|_t(A) \end{bmatrix}$$

is a completely positive map from $\mathfrak{A}(\mathfrak{H})$ into $M_2(\mathbb{C})$ for each $t > 0$ and, therefore, since μ, ν and τ are boundary weights, the limiting map defined for all $A \in \mathfrak{A}(\mathfrak{H})$ by

$$\psi_o(A) = \begin{bmatrix} \mu(A) & \kappa\tau(A) \\ \kappa\tau^*(A) & \nu(A) \end{bmatrix}$$

is completely positive. By Lemma 5.12, we have that $\kappa\gamma$ is an ordinary internal corner from ω to η .

Conversely, suppose $\kappa \geq 1$ and $\kappa\gamma$ is a corner from ω to η and

$$|h(t)| = \frac{(1 + \mu|_t(\Lambda(T_1)))^{\frac{1}{2}}(1 + \nu|_t(\Lambda(T_2)))^{\frac{1}{2}}}{|1 + \tau|_t(\Lambda(Q))|} \leq \kappa$$

for all $t > 0$. Then, in particular, for all $t > 0$, the scalar $1 + \tau|_t(\Lambda(Q))$ is not zero. Therefore, if $\Theta = \omega + \gamma + \gamma^* + \eta$, then its generalized boundary representation $\Pi_t^\#$ is well-defined and it is given by

$$\Pi_t^\#(A) = \begin{bmatrix} (1 + \mu|_t(\Lambda(T_1)))^{-1}\mu|_t(A)T_1 & (1 + \tau|_t(\Lambda(Q)))^{-1}\tau|_t(A)Q \\ (1 + \tau^*|_t(\Lambda(Q^*)))^{-1}\tau^*|_t(A)Q^* & (1 + \nu|_t(\Lambda(T_2)))^{-1}\nu|_t(A)T_2 \end{bmatrix}.$$

By an argument using Lemma 5.12 analogous to the one above, the map $\Pi_t^\#$ is completely positive if and only if the map

$$\psi_t(A) = \begin{bmatrix} \mu|_t(A) & |h(t)|\tau|_t(A) \\ |h(t)|\tau^*|_t(A) & \nu|_t(A) \end{bmatrix}$$

is completely positive. But this follows immediately from the fact that $\kappa\gamma$ is an internal corner from ω to η and $|h(t)| \leq \kappa$ for all $t > 0$. Therefore γ is an internal q -corner from ω to η .

Finally, we show $\kappa\gamma$ is a trivially maximal corner from ω to η if ω and η are unbounded. Suppose $\lambda > 1$ and $\lambda\kappa\gamma$ is a corner from ω to η . From the inequality for κ and the fact that $\mu|_t(\Lambda(T_1))$, $\nu|_t(\Lambda(T_2))$ and, thus, $|\tau|_t(\Lambda(Q))|$ tend to infinity as $t \rightarrow 0+$ we have

$$\lim_{t \rightarrow 0+} \frac{\mu|_t(\Lambda(T_1))\nu|_t(\Lambda(T_2))}{|\tau|_t(\Lambda(Q))|^2} \leq \kappa^2.$$

Since $\lambda\kappa\gamma$ is a corner we have $\lambda^2\kappa^2|\tau|_t(\Lambda(Q))|^2 \leq \mu|_t(\Lambda(T_1))\nu|_t(\Lambda(T_2))$ for $t > 0$ and, therefore,

$$\lim_{t \rightarrow 0+} \frac{\mu|_t(\Lambda(T_1))\nu|_t(\Lambda(T_2))}{|\tau|_t(\Lambda(Q))|^2} \geq \lambda^2\kappa^2 > \kappa^2.$$

This contradicts the previous limit inequality so $\kappa\gamma$ is a trivially maximal corner from ω to η . \square

The following generalizes Theorem 3.14 of [14] to q -pure range rank one q -weight maps, and it provides a useful criterion for the determining the existence of q -corners.

Theorem 5.14. *Suppose ω and η are completely orthogonal q -pure range rank one q -weight maps on a finite-dimensional Hilbert space \mathfrak{K} so $\omega(\rho)(A) = \rho(T_1)\mu(A)$ and $\eta(\rho)(A) = \rho(T_2)\nu(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$ where T_1 and T_2 are non-zero projections and μ and ν are q -pure q -weights on $\mathfrak{A}(\mathfrak{H})$. Then there is a non zero range rank one q -corner γ from ω to η if and only if there is a unitary operator $U \in \mathfrak{B}(\mathfrak{K})$ and a $z > 0$ so that $UT_1U^* = T_2$ and μ and ν can be expressed in the form*

$$\mu(A) = \sum_{k \in I} (f_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} f_k)$$

$$\nu(A) = \sum_{k \in I} (g_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} g_k)$$

for $A \in \mathfrak{A}(\mathfrak{H})$ and $g_k = z(U \otimes I)f_k + h_k$, with f_k in the range of $T_1 \otimes I$ and h_k in the range of $(I - \Lambda)^{\frac{1}{2}}(T_2 \otimes I)$ for $k \in I$ and

$$\sum_{k \in I} \|(I - \Lambda)^{-\frac{1}{2}} h_k\|^2 < \infty.$$

Proof. Before we begin the proof we remark that in the sums for ω and η we sum over the same index set I . Even though we sum over the same index set the sums for ω and η can have different numbers of non zero terms since some of the f_k 's or g_k 's can be zero.

Assume the setup and notation of the theorem. Let us assume that $g_k = z(U \otimes I)f_k + h_k$, with $z > 0$, f_k in the range of $T_1 \otimes I$, h_k in the range of $(I - \Lambda)^{1/2}(T_2 \otimes I)$ for each $k \in I$ and the sum involving the h_k 's given above converges. Since the sum involving the h_k 's above converges and $0 \leq \Lambda \leq I$ we have

$$r = \sum_{k \in I} \|\Lambda^{\frac{1}{2}}(I - \Lambda)^{-\frac{1}{2}} h_k\|^2 < \infty.$$

Let $\tau \in \mathfrak{A}(\mathfrak{H})_*$ be the boundary weight given by

$$\tau(A) = \sum_{k \in I} (f_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} g_k)$$

and let

$$\lambda = \frac{2z}{1 + z^2 + r}$$

We shall prove that the map $\gamma : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{A}(\mathfrak{H})_*$ given by

$$\gamma(\rho)(A) = \lambda \rho(T_1 U^*) \tau(A)$$

is an internal q -corner from ω to η . To that end, we will use Lemma 5.13. Notice that the operator $Q = T_1 U^*$ satisfies (26). Furthermore, γ is an ordinary internal corner from ω to η . Indeed, since $0 < \lambda \leq 1$, it suffices to prove that

$$h(t) = \frac{(1 + \mu|_t(\Lambda(T_1)))^{1/2} (1 + \nu|_t(\Lambda(T_2)))^{1/2}}{1 + \tau|_t(\Lambda(T_1 U^*))}$$

is well-defined, satisfies $|h(t)| \leq \lambda^{-1}$ for all $t > 0$ and $\lambda^{-1}\gamma$ is an ordinary internal corner from ω to η . Let $\Theta = \omega + \lambda^{-1}\gamma + \lambda^{-1}\gamma^* + \eta$, and notice that using the matricial identification from Remark 5.10 we have that

$$\check{\Theta}(A) = \begin{bmatrix} \mu(A)T_1 & \tau(A)T_1 U^* \\ \tau(A)UT_1 & \nu(A)T_2 \end{bmatrix}.$$

By Lemma 5.12, $\check{\Theta}$ is completely positive if and only if the map

$$\psi(A) = \begin{bmatrix} \mu(A) & \tau(A) \\ \tau(A) & \nu(A) \end{bmatrix}$$

is completely positive. But this follows immediately from observing that by the definition of μ , ν and τ , for all $B \in \mathfrak{B}(\mathfrak{H})$, if $A = (1 - \Lambda)^{1/2} B (1 - \Lambda)^{1/2}$,

$$\psi(A) = \sum_{k \in I} \begin{bmatrix} (f_k, B f_k) & (f_k, B g_k) \\ (g_k, B f_k) & (g_k, B g_k) \end{bmatrix}$$

hence ψ is a sum of completely positive maps. Finally, we shall prove that the function $h(t)$ defined in Lemma 5.13 is well-defined, bounded, and $\sup |h(t)| \leq \lambda^{-1}$. It suffices to show that

$$(29) \quad \lambda^2 (1 + \mu|_t(\Lambda(T_1))) (1 + \nu|_t(\Lambda(T_2))) \leq |1 + \lambda\tau|_t(\Lambda(T_1U^*))|^2$$

since by taking the limit as $t \rightarrow 0+$ we obtain

$$(30) \quad \lambda^2 (1 + \mu(\Lambda(T_1))) (1 + \nu(\Lambda(T_2))) \leq |1 + \lambda\tau(\Lambda(T_1U^*))|^2$$

This equation implies that

$$0 < \lambda \leq |1 + \lambda\tau(\Lambda(T_1U^*))|$$

hence $h(t)$ is well-defined for all $t > 0$. Furthermore, also by equation (30), we have that $|h(t)| \leq \lambda^{-1}$, for all $t > 0$. We proved above that $\lambda^{-1}\gamma$ is an ordinary internal corner, hence assuming that (29) holds then we have by Lemma 5.13 that γ is an internal q -corner.

Let us prove (29). By expanding it, we obtain

$$\begin{aligned} \lambda^2(1 + \mu|_t(\Lambda(T_1)) + \nu|_t(\Lambda(T_2)) + \mu|_t(\Lambda(T_1))\nu|_t(\Lambda(T_2)) - |\tau|_t(\Lambda(T_1U^*))|^2) \\ \leq 1 + 2\lambda Re(\tau|_t(\Lambda(T_1U^*))). \end{aligned}$$

To give this inequality a name so we can refer to it we will call this inequality the determinant inequality. We will need some notation. Let

$$\zeta(A) = \sum_{k \in I} (f_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} h_k)$$

and

$$\vartheta(A) = \sum_{k \in I} (h_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} h_k)$$

for $A \in \mathfrak{A}(\mathfrak{K})$. Then we have

$$\nu(A) = z^2 \mu((U^* \otimes I)A(U \otimes I)) + z\zeta((U^* \otimes I)A) + z\zeta^*(A(U \otimes I)) + \vartheta(A)$$

and

$$\zeta(A) = z\mu(A(U \otimes I)) + \vartheta(A)$$

for $A \in \mathfrak{A}(\mathfrak{H})$. Then we have

$$\nu|_t(\Lambda(T_2)) = z^2 \mu|_t(\Lambda(T_1)) + z\zeta|_t(\Lambda(T_1U^*)) + z\zeta^*|_t(\Lambda(UT_1)) + \vartheta|_t(\Lambda(T_2))$$

and

$$\zeta|_t(\Lambda(T_1U^*)) = z\mu|_t(\Lambda(T_1)) + \vartheta|_t(\Lambda(T_1U^*)).$$

To simplify the determinant inequality let

$$a = \mu|_t(\Lambda(T_1)), \quad b = \zeta|_t(\Lambda(T_1U^*)) \quad \text{and} \quad c = \vartheta|_t(\Lambda(T_2)).$$

Then the determinant inequality becomes

$$\lambda^2((1+a)(1+z^2a+zb+z\bar{b}+c) - |za+b|^2) \leq 1 + 2\lambda(az + Re(b))$$

and with a slight further simplification this becomes

$$\lambda^2(1+a+az^2+zb+z\bar{b}+c+ac-|b|^2) \leq 1 + 2\lambda(az + Re(b)).$$

Note that $c = \vartheta|_t(\Lambda(T_2)) \leq \vartheta(\Lambda(T_2)) = r$ so if we replace c by r the term on the left hand side of the above inequality does not decrease so if the above inequality is satisfied with c replaced by r it will be satisfied. Then to establish the determinant inequality it is sufficient to prove that

$$(31) \quad \lambda^2(1+a+az^2+2zRe(b)+r+ar-|b|^2) \leq 1 + 2\lambda(az + Re(b)).$$

Since $\lambda = 2z(1+z^2+r)^{-1}$ we have

$$(32) \quad \lambda^2(1+z^2+r) = 2\lambda z.$$

Multiplying both sides of this equation by a we find

$$(33) \quad \lambda^2(a+az^2+ar) = 2\lambda az.$$

Since $|\lambda(z-b)-1|^2 \geq 0$ we have

$$(34) \quad \lambda^2(-z^2+2zRe(b)-|b|^2) \leq -2\lambda z + 2\lambda Re(b) + 1.$$

If we add equations (32) and (33) to inequality (34) we obtain inequality (31). Hence, γ is a non-zero rank one internal q -corner from ω to η .

Now we give the proof in converse direction so we assume γ is a non-zero range rank one q -corner from ω to η . From Lemma 5.13, we know γ is of the form $\gamma(\rho)(A) = \rho(Q)\tau(A)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$ and $\|Q\| = 1$, $QQ^* \leq T_1$ and $Q^*Q \leq T_2$ and if

$$h(t) = \frac{(1 + \mu|_t(\Lambda(T_1)))^{\frac{1}{2}}(1 + \nu|_t(\Lambda(T_2)))^{\frac{1}{2}}}{1 + \tau|_t(\Lambda(Q))}$$

then there is a number $\kappa \geq 1$ so that $|h(t)| \leq \kappa$ for all $t > 0$ and $|h(t)| \rightarrow \kappa$ as $t \rightarrow 0+$ and $\kappa\gamma$ is an ordinary internal corner from ω to η .

We will show there is a constant K and complex number $x \neq 0$ and so that

$$(35) \quad |x|^2 \mu|_t(\Lambda(T_1)) + \nu|_t(\Lambda(T_2)) - 2\kappa \operatorname{Re}(\overline{x}\tau|_t(\Lambda(Q))) \leq K$$

By Remark 5.10 and Lemma 5.12, the map from $\mathfrak{A}(\mathfrak{H})$ into $M_2(\mathbb{C})$ given by

$$\psi_o(A) = \begin{bmatrix} \mu(A) & \kappa\tau(A) \\ \kappa\tau^*(A) & \nu(A) \end{bmatrix}$$

is completely positive. Now consider the family of matrices

$$M_t = \begin{bmatrix} 1 + \mu|_t(\Lambda(T_1)) & \kappa + \kappa\tau|_t(\Lambda(Q)) \\ \kappa + \kappa\tau^*|_t(\Lambda(Q^*)) & 1 + \nu|_t(\Lambda(T_2)) \end{bmatrix}$$

Note that if $0 < t < s$ then

$$M_t - M_s = \begin{bmatrix} \mu(\Lambda_t^s(T_1)) & \kappa\tau(\Lambda_t^s(Q)) \\ \kappa\tau^*(\Lambda_t^s(Q^*)) & \nu(\Lambda_t^s(T_2)) \end{bmatrix} = \psi_o(\Lambda_t^s(T_1 + Q + Q^* + T_2))$$

where $\Lambda_t^s(A) = E(t, s)\Lambda(A)E(t, s)$. Since ψ_o is completely positive and Q satisfies (26), it follows that $M_t - M_s$ is a positive matrix. Since $|h(t)| \leq \kappa$ for all $t > 0$ we have

$$(1 + \mu|_t(\Lambda(T_1)))(1 + \nu|_t(\Lambda(T_2))) \leq |\kappa + \kappa\tau|_t(\Lambda(Q))|^2.$$

This tells us the determinant of M_t is less than or equal to zero so for each $t > 0$ there is a unit vector $v(t)$ so that $M_t v(t) = \lambda(t)v(t)$ and $\lambda(t) \leq 0$. We will prove there is a unit vector v so that $(v, M_t v) \leq 0$ for all $t > 0$.

Since the set of unit vectors in \mathbb{C}^2 is compact in the norm topology there is at least one accumulation point of v_t as $t \rightarrow 0+$. Let v be such an accumulation point so for each $\varepsilon > 0$ there is a $t \in (0, \varepsilon)$ with $\|v - v_t\| < \varepsilon$. We show $(v, M_t v) \leq 0$ for all $t > 0$. Suppose $t > 0$ and $\varepsilon > 0$. Let $\varepsilon_1 = \min(\varepsilon, \frac{1}{2}\varepsilon/\|M_t\|)$. Then there is an $s \in (0, \varepsilon_1)$ with $\|v - v_s\| < \varepsilon_1$. We have

$$(v, M_t v) = (v, M_t v) - (v_s, M_t v_s) + (v_s, M_t v_s) - (v_s, M_s v_s) + (v_s, M_s v_s).$$

We have

$$\begin{aligned} (v, M_t v) - (v_s, M_t v_s) &= ((v - v_s), M_t v) + (v_s, M_t(v - v_s)) \\ &\leq 2\|v - v_s\| \|M_t\| < 2\varepsilon_1 \|M_t\| \leq \varepsilon. \end{aligned}$$

Since M_t is non increasing and $0 < s < t$ we have

$$(v_s, M_t v_s) - (v_s, M_s v_s) \leq 0.$$

And we have

$$(v_s, M_s v_s) = \lambda(s) \leq 0.$$

Combining the previous four relation we find $(v, M_t v) \leq \varepsilon$ and since ε is arbitrary we have $(v, M_t v) \leq 0$ for all t .

Notice that the non-zero vector v so that $(v, M_t v) \leq 0$ for all $t > 0$ cannot be a multiple of $(1, 0)$ or $(0, 1)$ because in the first case we have $(v, M_t v) = 1 + \mu|_t(\Lambda(T_1))$ and the second case $(v, M_t v) = 1 + \nu|_t(\Lambda(T_2))$ and neither of these are less than or equal to zero. So the vector v must be a multiple of a vector of the form $w = (x, -1)$ with $x \neq 0$. Then we have $(w, M_t w) \leq 0$ which yields the promised inequality (35)

$$|x|^2 \mu|_t(\Lambda(T_1)) + \nu|_t(\Lambda(T_1)) - 2\kappa \operatorname{Re}(\overline{x}\tau|_t(\Lambda(Q))) \leq 2\kappa \operatorname{Re}(x) - 1 - |x|^2 = K$$

for all $t > 0$.

The fact that ψ_o is completely positive on $\mathfrak{A}(\mathfrak{H})$ means the mapping $\phi(A) = \psi_o((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}})$ is a completely positive map from $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathbb{C} \oplus \mathbb{C})$ and every such completely positive map is of the form

$$\phi(A) = \sum_{i \in I} S_i A S_i^*$$

where the S_i are linear maps from \mathfrak{H} to $\mathbb{C} \oplus \mathbb{C}$. Since every such map is specified by a pair of vectors $f_i, g_i \in \mathfrak{H}$ it follows that

$$(36) \quad \mu(A) = \sum_{i \in I} (f_i, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} f_i)$$

$$(37) \quad \nu(A) = \sum_{i \in I} (g_i, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} g_i)$$

$$(38) \quad \kappa\tau(A) = \sum_{i \in I} (f_i, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} g_i)$$

for $A \in \mathfrak{A}(\mathfrak{H})$. Furthermore, since ω and η are completely orthogonal with respect to T_1 and T_2 and γ is an internal corner from ω to η , by replacing f_i by $(T_1 \otimes I)f_i$ and replacing g_i by $(T_2 \otimes I)g_i$, we still have that (36), (37) and (38) hold.

Then from inequality (35) we have

$$(39) \quad \sum_{i \in I} |x|^2 (f_i^t, (T_1 \otimes I)f_i^t) + (g_i^t, (T_2 \otimes I)g_i^t) - 2\operatorname{Re}(x f_i^t, (Q \otimes I)g_i^t) \leq K$$

where $f_i^t = \Lambda^{\frac{1}{2}}(I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i$ and $g_i^t = \Lambda^{\frac{1}{2}}(I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i$ for $t > 0$ and $i \in I$. We can write the above sum in the form

$$\sum_{i \in I} \|x(T_1 \otimes I)f_i^t - (Q \otimes I)g_i^t\|^2 + (g_i^t, [(T_2 - Q^*Q) \otimes I]g_i^t) \leq K$$

From this it follows that $\nu|_t(\Lambda(T_2 - Q^*Q)) \leq K$ for all $t > 0$. Now if $T_2 \neq Q^*Q$ there is a rank one projection $e \in \mathfrak{B}(\mathfrak{H})$ and a real number $y > 0$ that $ye \leq T_2 - Q^*Q$ from which it follows that $\nu|_t(\Lambda(e)) \leq K/y$ for all $t > 0$. But since η is q -pure it follows from Theorem 5.7 that $\nu(\Lambda(e)) = \infty$. Hence, we conclude that $T_2 = Q^*Q$.

The sum (39) can also be written in the form

$$(40) \quad \sum_{i \in I} \|x(Q^* \otimes I)f_i^t - (T_2 \otimes I)g_i^t\|^2 + |x|^2 (f_i^t, [(T_1 - QQ^*) \otimes I]f_i^t) \leq K$$

From which we conclude $\mu|_t(\Lambda(T_1 - QQ^*)) \leq K|x|^{-2}$ for all $t > 0$. Since ω is q -pure we conclude by repeating the argument above that $T_1 = QQ^*$. Hence, Q is a partial isometry from the $\operatorname{Range}(T_2)$ to the $\operatorname{Range}(T_1)$. We construct number z and the unitary U as promised in the statement of the theorem. (Strictly speaking, we are assuming that $\dim \mathfrak{H} < \infty$, hence the argument is straightforward, however we describe a proof which works even when $\dim \mathfrak{H} = \infty$ anticipating future considerations). Let $z = |x|$ and $u = x/|x|$ so $x = zu$. Since $Q^*Q = T_2$ and $QQ^* = T_1$ it follows that the ranges of T_1 and T_2 have the same dimension. If T_1 is of finite rank then the $\operatorname{Range}(T_1)^\perp$ and $\operatorname{Range}(T_2)^\perp$ have the same dimension and if T_1 is of infinite rank then since T_1 and T_2 are disjoint it follows that $\operatorname{Range}(T_1)^\perp$ and $\operatorname{Range}(T_2)^\perp$ are both of infinite dimension. So $\operatorname{Range}(T_1)^\perp$ and $\operatorname{Range}(T_2)^\perp$ have the same dimension. We define U as follows. We define U on $\operatorname{Range}(T_1)$ as $UT_1 = uQ^*$, and since $\operatorname{Range}(T_1)^\perp$ and $\operatorname{Range}(T_2)^\perp$ have the same dimension we can define a partial isometry S from $\operatorname{Range}(I - T_1)$ onto $\operatorname{Range}(I - T_2)$. Then we define U on $\operatorname{Range}(I - T_1)$ as $U(I - T_1) = S(I - T_1)$. Then we have U is unitary and $UT_1U^* = T_2$ and we have from inequality (40) that

$$(41) \quad \sum_{i \in I} \|z(UT_1 \otimes I)f_i^t - (T_2 \otimes I)g_i^t\|^2 \leq K$$

for all $t > 0$. This is the important estimate which will yield the estimate stated in the theorem when we use the normalization properties of μ and ν . In subsequent calculation it is useful to have the

formulae

$$x = zu, \quad z = |x|, \quad T_2 U = U T_1 = u Q^* \quad \text{and} \quad \overline{u} Q = T_1 U^* = U^* T_2$$

Now expressing inequality (41) inner product form we have

$$(42) \quad \sum_{i \in I} |x|^2 ((I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i, \Lambda(T_1) (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i)$$

$$(43) \quad - \sum_{i \in I} 2 \operatorname{Re}(x (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i, \Lambda(Q) (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i)$$

$$(44) \quad + \sum_{i \in I} ((I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i, \Lambda(T_2) (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i) \leq K$$

We also have

$$(45) \quad \mu|_t(I - \Lambda(T_1)) = \sum_{i \in I} ((I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i, (I - \Lambda(T_1)) (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i) \leq 1$$

and

$$(46) \quad \nu|_t(I - \Lambda(T_2)) = \sum_{i \in I} ((I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i, (I - \Lambda(T_2)) (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i) \leq 1$$

We will need one more inequality. Notice that if $A = T_1 + Q + Q^* + T_2$, then $B = (A \otimes I) - \Lambda(A)$ is positive. Therefore, from the complete positivity of ψ_o , and using the fact that $\mu(B|_t) = \mu|_t(I - \Lambda(T_1))$, $\nu(B|_t) = \nu|_t(I - \Lambda(T_2))$ and $\tau(B|_t) = \tau|_t((uU^* \otimes I) - \Lambda(Q))$, we have that

$$0 \leq \psi_o(B|_t) = \begin{bmatrix} \mu|_t(I - \Lambda(T_1)) & \kappa\tau|_t(uU^* \otimes I - \Lambda(Q)) \\ \kappa\tau^*|_t(\overline{u}U \otimes I - \Lambda(Q)) & \nu|_t(I - \Lambda(T_2)) \end{bmatrix}$$

whence by considering the determinant we obtain

$$|\kappa\tau|_t(uU^* \otimes I - \Lambda(Q))|^2 \leq \mu|_t(I - \Lambda(T_1))\nu|_t(I - \Lambda(T_2)) \leq 1 \cdot 1 = 1.$$

This gives us the inequality

$$(47) \quad - \sum_{i \in I} 2 \operatorname{Re}(x (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i, (uU^* \otimes I - \Lambda(Q)) (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i) \leq 2|x|.$$

Combining inequalities (42) through (47), we obtain

$$\begin{aligned} & \sum_{i \in I} |x|^2 ((I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i, (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i) \\ & - \sum_{i \in I} 2 \operatorname{Re}(x (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) f_i, (uU^* \otimes I) (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i) \\ & + \sum_{i \in I} ((I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i, (I - \Lambda)^{-\frac{1}{2}} E(t, \infty) g_i) \leq K + 1 + 2z + z^2 \end{aligned}$$

And the three sums collapse into a simpler sum

$$\sum_{i \in I} \|(I - \Lambda)^{-\frac{1}{2}} E(t, \infty) h_i\|^2 \leq K + (1 + z)^2$$

where $h_i = z(U \otimes I) f_i - g_i$ for $i \in I$. Since the sum is independent of t we have $h_i \in \operatorname{Range}((I - \Lambda)^{\frac{1}{2}}) = \operatorname{Domain of } (I - \Lambda)^{-\frac{1}{2}}$ and

$$\sum_{i \in I} \|(I - \Lambda)^{-\frac{1}{2}} h_i\|^2 \leq K + (1 + z)^2.$$

This completes the proof of the theorem. \square

In the case when ω and η have trivial normal spines, by Proposition 5.11 and Theorem 5.8, the only non-zero q -corners between them have to be range rank one. If they are unital, this leads to the following immediate corollary.

Corollary 5.15. *For $j = 1, 2$, let ω_j be a unital q -pure range rank one q -weight map over \mathfrak{K}_j finite-dimensional with trivial normal spine. If $\dim \mathfrak{K}_1 \neq \dim \mathfrak{K}_2$, then ω_1 and ω_2 induce E_0 -semigroups that are not cocycle conjugate.*

6. THE RANGE RANK TWO CASE

In this section we analyze range rank two q -weights. This means that ω is of the form

$$\omega(\rho)(A) = \rho(e_1)\mu_1(A) + \rho(e_2)\mu_2(A)$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$. Typically we assume that e_1 and e_2 are positive norm one operators of the following particular form.

Proposition 6.1. *Let \mathfrak{K} be a separable Hilbert space and let $\omega : \mathfrak{B}(\mathfrak{K})_* \rightarrow \mathfrak{B}(\mathfrak{H})_*$ be a unital q -weight map with range rank 2 and trivial normal spine. Then there exist positive $\mu_1, \mu_2 \in \mathfrak{A}(\mathfrak{H})_*$ and $e_1, e_2 \in \mathfrak{B}(\mathfrak{K})$ positive operators with norm one such that for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$,*

$$\omega(\rho)(A) = \rho(e_1)\mu_1(A) + \rho(e_2)\mu_2(A).$$

Furthermore, given $x_1, x_2 \in \mathbb{R}$, we have that $0 \leq x_1 e_1 + x_2 e_2 \leq I$ if and only if $x_1, x_2 \in [0, 1]$.

Proof. Since ω has a trivial normal spine, by Theorem 3.2 there exists $L : \mathfrak{B}(\mathfrak{K}) \rightarrow \mathfrak{B}(\mathfrak{K})$ a boundary expectation for ω . Since $\text{Range}(L) = \text{Range}(\tilde{\omega})$, we have that the Choi-Effros algebra \mathfrak{R}_L is isomorphic to $\mathbb{C} \oplus \mathbb{C}$. Furthermore, notice that the unit of \mathfrak{R}_L is given by $L(I)$. Let $e_1 \in \mathfrak{R}_L$ be a minimal central projection (with respect to the Choi-Effros multiplication), and notice that $e_1 \neq 0$ and $e_1 \neq L(I)$. Let $e_2 = L(I) - e_1$, also a minimal projection. Then we have that for all $X \in \text{Range}(L)$,

$$X = e_1 \otimes X \otimes e_1 + e_2 \otimes X \otimes e_2 = L(e_1 X e_1) + L(e_2 X e_2) = \nu_1(X)e_1 + \nu_2(X)e_2$$

for some states ν_1, ν_2 on \mathfrak{R}_L since e_1 and e_2 are minimal projections in \mathfrak{R}_L . Thus, for all $A \in \mathfrak{A}(\mathfrak{H})$,

$$\tilde{\omega}(A) = \nu_1(\tilde{\omega}(A))e_1 + \nu_2(\tilde{\omega}(A))e_2$$

Now let $\mu_j = \nu_j \circ \tilde{\omega}$ for $j = 1, 2$. Those are clearly positive boundary weights.

The final part of the statement follows immediately from the observation that e_1 and e_2 are orthogonal projections in the algebra \mathfrak{R}_L such that $e_1 + e_2 = I_{\mathfrak{R}_L} = L(I)$ and $L(I) = I$ since ω is unital. Indeed, $\tilde{\omega}(I - \Lambda) = I$ and L fixes the range of $\tilde{\omega}$. \square

We recall the following notation. If $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$, then we define the operator of $\mathfrak{B}(\mathfrak{H})$

$$A|_t = E(t, \infty)AE(t, \infty).$$

We emphasize that in fact, for all $t > 0$, we have that $A|_t \in \mathfrak{A}(\mathfrak{H})$.

Theorem 6.2. *Let \mathfrak{K} be a separable Hilbert space. Suppose e_1 and e_2 are two positive operators in $\mathfrak{B}(\mathfrak{K})$ so that if x_1 and x_2 are real numbers then*

$$(48) \quad 0 \leq x_1 e_1 + x_2 e_2 \leq I \quad \Longleftrightarrow \quad x_1, x_2 \in [0, 1].$$

Let ω be a range rank two boundary weight map of the form

$$(49) \quad \omega(\rho)(A) = \rho(e_1)\mu_1(A) + \rho(e_2)\mu_2(A)$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$ where μ_1 and μ_2 are positive boundary weights on $\mathfrak{A}(\mathfrak{H})$. Let h_1 and h_2 be functions

$$(50) \quad h_1(t) = \frac{\mu_1(\Lambda(e_2)|_t)}{1 + \mu_2(\Lambda(e_2)|_t)} \quad \text{and} \quad h_2(t) = \frac{\mu_2(\Lambda(e_1)|_t)}{1 + \mu_1(\Lambda(e_1)|_t)}$$

defined for $t > 0$. Let

$$(51) \quad \kappa_1 = \sup(h_1(t) : t > 0) \quad \text{and} \quad \kappa_2 = \sup(h_2(t) : t > 0)$$

Then ω is a q -weight if and only if the following conditions are satisfied. The numbers κ_1 and κ_2 are in the closed interval $[0, 1]$ and the weights μ_1 and μ_2 satisfy the weight inequalities

$$(52) \quad \mu_1 \geq \kappa_1 \mu_2 \quad \text{and} \quad \mu_2 \geq \kappa_2 \mu_1$$

and the numbers $x = \mu_1(I - \Lambda(e_1 + e_2))$ and $y = \mu_2(I - \Lambda(e_1 + e_2))$ are in the set S_o consisting of a parallelogram in the (x, y) plane with opposite vertices $(0, 0)$ and $(1, 1)$ and whose sides are parallel to

the lines $x = \kappa_1 y$ and $y = \kappa_2 x$. In the event that either $\kappa_1 = 1$ or $\kappa_2 = 1$ then the set S_o consists of the line segment $0 \leq x = y \leq 1$. If both $\kappa_1 < 1$ and $\kappa_2 < 1$ then set S_o consists of the points satisfying the inequalities

$$0 \leq x - \kappa_1 y \leq 1 - \kappa_1 \quad \text{and} \quad 0 \leq y - \kappa_2 x \leq 1 - \kappa_2.$$

Furthermore, in the event μ_1 and μ_2 satisfy these conditions then the functions $h_1(t)$ and $h_2(t)$ are non increasing and $h_1(t) \rightarrow \kappa_1$ and $h_2(t) \rightarrow \kappa_2$ as $t \rightarrow 0+$ and if either $\kappa_1 = 1$ or $\kappa_2 = 1$ then $\mu_1 = \mu_2$ so ω is a range rank one q -weight.

Proof. Before we begin the proof we will establish some notation and basic facts that we will use through out the proof. Let $e_1, e_2 \in \mathfrak{B}(\mathfrak{K})$ be positive operators satisfying (48), let μ_1 and μ_2 be positive boundary weights on $\mathfrak{A}(\mathfrak{H})$ and suppose that ω is a boundary weight map of the form (49). Then ω is a completely positive boundary weight map and its dualized boundary weight map $\check{\omega}$ is well-defined. Let $X(t)$ for $t > 0$ denote the matrix representing the restriction of $\check{\omega}|_t \Lambda$ to the invariant subspace of $\mathfrak{B}(\mathfrak{K})$ spanned by e_1, e_2 :

$$(53) \quad X(t) = \begin{bmatrix} \mu_1(\Lambda(e_1)|_t) & \mu_1(\Lambda(e_2)|_t) \\ \mu_2(\Lambda(e_1)|_t) & \mu_2(\Lambda(e_2)|_t) \end{bmatrix}.$$

We denote the entries of $X(t)$ by $x_{ij}(t)$ for $i, j = 1, 2$. Sometimes in calculations we will write simply X or x_{ij} instead of $X(t)$ or $x_{ij}(t)$. When we do this we of course mean that all terms in the equation are to be evaluated at the same t .

Notice that $I + \check{\omega}|_t \Lambda$ is invertible if and only if it is injective. Indeed, $\check{\omega}|_t \Lambda$ is finite rank, therefore $I + \check{\omega}|_t \Lambda$ is a Fredholm operator of index zero. Therefore, if it is injective then it is automatically surjective therefore invertible. Let us describe a convenient necessary and sufficient condition for its injectivity. Since the range of $\check{\omega}|_t$ is an invariant subspace under $I + \check{\omega}|_t \Lambda$, the same must hold for $(I + \check{\omega}|_t \Lambda)^{-1}$ whenever the inverse exists. Observe that if $T : \mathfrak{B}(\mathfrak{K}) \rightarrow \mathfrak{B}(\mathfrak{K})$ is any operator, then $\ker(I + T) \subseteq \text{Range}(T)$, hence we have that $I + \check{\omega}|_t \Lambda$ is injective if and only if its restriction to the range of $\check{\omega}|_t$ is injective. Thus $I + \check{\omega}|_t \Lambda$ is invertible if and only if $\det(I + X(t)) \neq 0$.

Let us assume that $\det(I + X(t)) \neq 0$ for all $t > 0$. In this case, the generalized boundary representation for ω can be obtained by solving the equation $(I + X(t))\pi_t^\# = \check{\omega}|_t$, from which we obtain that

$$\pi_t^\#(A) = \ell_t^{(1)}(A)e_1 + \ell_t^{(2)}(A)e_2, \quad \forall A \in \mathfrak{B}(\mathfrak{H})$$

where

$$(54) \quad \begin{bmatrix} \ell_t^{(1)}(A) \\ \ell_t^{(2)}(A) \end{bmatrix} = \frac{1}{\det(I + X(t))} \begin{bmatrix} 1 + x_{22}(t) & -x_{12}(t) \\ -x_{21}(t) & 1 + x_{11}(t) \end{bmatrix} \begin{bmatrix} \mu_1|_t(A) \\ \mu_2|_t(A) \end{bmatrix}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. Furthermore, recall that ω is q -weight if and only if $\pi_t^\#$ is a completely positive contraction for all $t > 0$. But observe that by equation (48), we will have that $\pi_t^\#$ is positive if and only if both $\ell_t^{(1)}$ and $\ell_t^{(2)}$ are positive linear functionals. However, if those linear functionals are positive, then clearly $\pi_t^\#$ is completely positive since it is the sum of completely positive maps. Furthermore, when $\pi_t^\#$ is positive, we have also by (48) that $\pi_t^\#$ is a contraction if and only if the map from $\mathfrak{B}(\mathfrak{H})$ to $\mathbb{C} \oplus \mathbb{C}$ given by (54) is a contraction. Thus, in order to simplify matters, we will abuse notation and denote also by $\pi_t^\#$ the map from $\mathfrak{B}(\mathfrak{H})$ to $\mathbb{C} \oplus \mathbb{C}$ given by

$$(55) \quad \pi_t^\#(A) = \begin{bmatrix} \ell_t^{(1)}(A) \\ \ell_t^{(2)}(A) \end{bmatrix}.$$

In summary, we conclude that the true generalized boundary representation is a completely positive contraction if and only if the map $\pi_t^\#$ given by (55) is a positive contraction. For the remainder of the proof we will refer to $\pi_t^\#$ given by (55) as if it were the true generalized boundary representation. Although this is not strictly true the conclusions we draw for it are still valid for the reasons we have given.

Notice in terms of the $x_{ij}(t)$ the functions h_1 and h_2 are

$$(56) \quad h_1(t) = \frac{x_{12}(t)}{1 + x_{22}(t)} \quad \text{and} \quad h_2(t) = \frac{x_{21}(t)}{1 + x_{11}(t)}$$

Our goal is to show that the conditions stated in the theorem are necessary and sufficient for ω to be a q -weight.

For the forward direction of the proof of the theorem we assume that ω is a q -weight map, which is equivalent to assuming $\pi_t^\#$ given above is a positive contraction for all $t > 0$. Notice that since ω is a q -weight and thus $\omega(\rho)(I - \Lambda) \leq \rho(I)$ for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$, we have that

$$\tilde{\omega}(I - \Lambda) = \mu_1(I - \Lambda)e_1 + \mu_2(I - \Lambda)e_2 \leq I$$

hence by (48), we have that $\mu_j(I - \Lambda) \leq 1$ for $j = 1, 2$, hence we find ourselves precisely in the framework described in the preamble of the current proof, and we have that $\det(I + X(t)) \neq 0$ for all $t > 0$.

The first problem we face is that we need to know the sign of the $\det(I + X(t))$. As we will see this determinant is never less than one. Recall that if ν is any boundary weight, then for all $A \in \mathfrak{B}(\mathfrak{H})$, we have that for all $t > 1$, $\nu|_t(A) = \nu|_1(A|_t)$ and $\nu|_1$ is normal, hence $\nu|_t(A) \rightarrow 0$ as $t \rightarrow \infty$. Therefore $x_{ij}(t) \rightarrow 0$ since as $t \rightarrow \infty$ so $\det(I + X(t))$ approaches one in the limit. Hence, there is a number $t_o > 0$ so that the determinant is greater than one half for $t > t_o$. Then for $t > t_o$

$$\begin{aligned} \pi_t^\#(\Lambda(e_1)) &\geq 0 && \text{yields} && x_{11}(t) + \det(X(t)) \geq 0, \\ \pi_t^\#(\Lambda(e_2)) &\geq 0 && \text{yields} && x_{22}(t) + \det(X(t)) \geq 0. \end{aligned}$$

Since

$$\det(I + X(t)) = 1 + x_{11}(t) + x_{22}(t) + \det(X(t)),$$

we find

$$\det(I + X(t)) \geq 1 + \max(x_{11}(t), x_{22}(t)) \geq 1$$

Hence, we have $\det(I + X(t)) \geq 1$ for $t > t_o$. Since this estimate is independent of t and the determinant is continuous in t we have the determinant is greater than one for all $t > 0$. Having established this fact we will use it in subsequent computations without comment.

Now making use of the fact that $\pi_t^\#$ is a contraction we have

$$\pi_t^\#(\Lambda(e_1 + e_2)) \leq 1$$

which yields

$$(57) \quad x_{12}(t) \leq 1 + x_{22}(t) \quad \text{and} \quad x_{21}(t) \leq 1 + x_{11}(t)$$

for $t > 0$. This then shows that $h_1(t) \leq 1$ and $h_2(t) \leq 1$ for all positive t and so we have $\kappa_1 \leq 1$ and $\kappa_2 \leq 1$.

The positivity of $\pi_t^\#$ yields the inequalities

$$(58) \quad \mu_1|_t \geq \frac{x_{12}(t)}{1 + x_{22}(t)} \mu_2|_t = h_1(t) \mu_2|_t$$

$$(59) \quad \mu_2|_t \geq \frac{x_{21}(t)}{1 + x_{11}(t)} \mu_1|_t = h_2(t) \mu_1|_t$$

for $t > 0$. We claim h_1 and h_2 are non increasing functions of t . Suppose $0 < t < s$. Applying the above inequalities to the positive elements $\Lambda(e_1)|_t - \Lambda(e_1)|_s$ and $\Lambda(e_2)|_t - \Lambda(e_2)|_s$ we find

$$\begin{aligned} x_{11}(t) - x_{11}(s) &\geq \frac{x_{12}(t)}{1 + x_{22}(t)} (x_{21}(t) - x_{21}(s)) \\ x_{12}(t) - x_{12}(s) &\geq \frac{x_{12}(t)}{1 + x_{22}(t)} (x_{22}(t) - x_{22}(s)) \\ x_{21}(t) - x_{21}(s) &\geq \frac{x_{21}(t)}{1 + x_{11}(t)} (x_{11}(t) - x_{11}(s)) \\ x_{22}(t) - x_{22}(s) &\geq \frac{x_{21}(t)}{1 + x_{11}(t)} (x_{12}(t) - x_{12}(s)) \end{aligned}$$

Multiplying by the denominators in the middle two inequalities we find

$$\begin{aligned} x_{12}(t) + x_{12}(t)x_{22}(s) &\geq x_{12}(s) + x_{22}(t)x_{12}(s) \\ x_{21}(t) + x_{21}(t)x_{11}(s) &\geq x_{21}(s) + x_{11}(t)x_{21}(s) \end{aligned}$$

which yields

$$h_1(t) \geq h_1(s) \quad \text{and} \quad h_2(t) \geq h_2(s)$$

for $0 < t < s$. Hence, h_1 and h_2 are non increasing functions. Since $h_1(t) \leq 1$ and $h_2(t) \leq 1$ for all $t > 0$ these functions have limits as $t \rightarrow 0+$. Hence, we have

$$h_1(t) \rightarrow \kappa_1 \leq 1 \quad \text{and} \quad h_2(t) \rightarrow \kappa_2 \leq 1$$

From inequalities (58) and (59), we see that the inequalities (52) hold.

Now we tackle the normalization conditions on the point in the xy -plane given by

$$x_o = \mu_1(I - \Lambda(e_1 + e_2)) \quad \text{and} \quad y_o = \mu_1(I - \Lambda(e_1 + e_2)).$$

Since, $\pi_t^\#$ is a completely positive contraction we have $\pi_t^\#(I)$ is a positive contraction for all $t > 0$ where $\pi_t^\#(I)$ is given by

$$\pi_t^\#(I) = \frac{1}{\det(I + X(t))} \begin{bmatrix} 1 + x_{22}(t) & -x_{12}(t) \\ -x_{21}(t) & 1 + x_{11}(t) \end{bmatrix} \begin{bmatrix} \mu_1(I|_t) \\ \mu_2(I|_t) \end{bmatrix}$$

We see $\pi_t^\#$ is a contraction if and only if the first and second entries of $\pi_t^\#$ are contractions by (48). Beginning with the top entry we have

$$(1 + x_{22}(t))\mu_1(I|_t) - x_{12}(t)\mu_2(I|_t) \leq (1 + x_{11}(t))(1 + x_{22}(t)) - x_{12}(t)x_{21}(t)$$

for $t > 0$. Now we have $I|_t = (I - \Lambda(e_1 + e_2))|_t + \Lambda(e_1 + e_2)|_t$ and putting this in the above inequality we have

$$\begin{aligned} ((1 + x_{22})\mu_1 - x_{12}\mu_2)((I - \Lambda(e_1 + e_2))|_t) + (1 + x_{22})(x_{11} + x_{12}) - x_{12}(x_{21} + x_{22}) \\ \leq (1 + x_{11})(1 + x_{22}) - x_{12}x_{21} \end{aligned}$$

Canceling terms we see this inequality is equivalent to

$$((1 + x_{22}(t))\mu_1 - x_{12}(t)\mu_2)((I - \Lambda(e_1 + e_2))|_t) \leq 1 + x_{22}(t) - x_{12}(t)$$

Now dividing by $(1 + x_{22}(t))$ we find

$$(60) \quad (\mu_1 - h_1(t)\mu_2)((I - \Lambda(e_1 + e_2))|_t) \leq 1 - h_1(t)$$

By symmetry we see the condition that the bottom term in $\pi_t^\#$ be a contraction is

$$(61) \quad (\mu_2 - h_2(t)\mu_1)((I - \Lambda(e_1 + e_2))|_t) \leq 1 - h_2(t)$$

Now we show that if $h_1(t) = 1$ or $h_2(t) = 1$ for some $t > 0$ then $\mu_1 = \mu_2$. Suppose for example $h_2(t_o) = 1$ for $t_o > 0$. It follows that $\kappa_2 = 1$ and, therefore, $\mu_2 \geq \mu_1$ by (52). Since h_1 is non increasing we have $h_1(t) = 1$ for all $t \in (0, t_o]$. Hence, by (61) we have

$$(\mu_2 - \mu_1)((I - \Lambda(e_1 + e_2))|_t) = 0$$

for all $t \in (0, t_o)$. But this means $(\mu_2 - \mu_1)(I - \Lambda(e_1 + e_2)) = 0$ and therefore, $0 \leq (\mu_2 - \mu_1)(I - \Lambda) \leq (\mu_2 - \mu_1)(I - \Lambda(e_1 + e_2)) = 0$. Hence, $\mu_2 - \mu_1$ is a positive weight which gives zero when applied to $I - \Lambda$ so $\mu_2 - \mu_1 = 0$. In the case where $\mu_1 = \mu_2$ we have $x_{11}(t) = x_{21}(t)$ and $x_{12}(t) = x_{22}(t)$ and $\pi_t^\#(I)$ is

$$\pi_t^\#(I) = \frac{1}{1 + \mu(\Lambda(e_1 + e_2))|_t} \begin{bmatrix} \mu(I|_t) \\ \mu(I|_t) \end{bmatrix}$$

for $t > 0$ where $\mu = \mu_1 = \mu_2$. Then $\pi_t^\#$ is a contraction if and only if

$$\mu((I - \Lambda(e_1 + e_2))|_t) \leq 1$$

for all $t > 0$ and this is equivalent to $\mu(I - \Lambda(e_1 + e_2)) \leq 1$. Hence, we see that if $h_2(t) = 1$ for some $t > 0$ then $\mu_1(I - \Lambda(e_1 + e_2))$ and $\mu_2(I - \Lambda(e_1 + e_2))$ satisfy the normalization condition given in the statement of the theorem. The same argument with the indices 1 and 2 interchanged shows that if $h_1(t) = 1$ for some $t > 0$ then $\mu_1 = \mu_2$ and the normalization conditions on μ_1 and μ_2 stated in the theorem are satisfied.

Now that we have taken care of the case where either $h_1(t) = 1$ or $h_2(t) = 1$ for some $t > 0$ we now assume that both $h_1(t) < 1$ and $h_2(t) < 1$ for all $t > 0$. We will show that if $\kappa_1 = 1$ then the normalization conditions of the theorem are satisfied. Suppose $\kappa_1 = 1$. Then inequality (60) becomes

$$(\mu_1 - \mu_2)((I - \Lambda(e_1 + e_2))|_t) + (1 - h_1(t))\mu_2((I - \Lambda(e_1 + e_2))|_t) \leq 1 - h_1(t)$$

Since $\mu_1 - \mu_2 \geq 0$ and $h_1(t) < 1$ for all $t > 0$ we have

$$\mu_2((I - \Lambda(e_1 + e_2))|_t) \leq 1$$

for all $t > 0$ and we conclude $\mu_2(I - \Lambda(e_1 + e_2)) \leq 1$. Then we conclude from the above inequalities that

$$(\mu_1 - \mu_2)((I - \Lambda(e_1 + e_2))|_t) \leq (1 - h_1(t))(1 - \mu_2((I - \Lambda(e_1 + e_2))|_t))$$

Since the right hand side of the above inequality converges to zero as $t \rightarrow 0+$ we conclude that $0 \leq (\mu_1 - \mu_2)((I - \Lambda(e_1 + e_2))) \leq 0$ so $\mu_1 = \mu_2$ and as we have seen this then leads to the conclusion that μ_1 and μ_2 satisfy the normalization condition in the statement of the theorem. The same argument with the indices 1 and 2 interchanged shows that if $\kappa_2 = 1$ then $\mu_2 = \mu_1$ and the normalization conditions are satisfied.

Thus to complete the proof of the forward direction of the theorem, all that remains is the case where both $\kappa_1 < 1$ and $\kappa_2 < 1$. We assume this to be the case. Then inequality (60) and (52) imply that for $t > 0$,

$$0 \leq (\mu_1 - \kappa_1\mu_2)((I - \Lambda(e_1 + e_2))|_t) \leq 1 - h_1(t),$$

from which we conclude that

$$0 \leq (\mu_1 - \kappa_1\mu_2)((I - \Lambda(e_1 + e_2))) \leq 1 - \kappa_1.$$

The same argument with the indices 1 and 2 interchanged shows

$$0 \leq (\mu_2 - \kappa_2\mu_1)((I - \Lambda(e_1 + e_2))) \leq 1 - \kappa_2.$$

These inequalities show that $\mu_1((I - \Lambda(e_1 + e_2)))$ and $\mu_2((I - \Lambda(e_1 + e_2)))$ lie in the parallelogram given in the statement of the theorem. Hence, we have shown if ω is a q -weight of the form (49) then μ_1 and μ_2 satisfy the conditions given in the statement of the theorem.

Now we prove the backward direction of the theorem. Let us assume that ω is of the form (49) and μ_1 and μ_2 satisfy the conditions given in the statement of the theorem. Notice that we automatically have that $\mu_j(I - \Lambda) \leq \mu_j(I - \Lambda(e_1 + e_2)) \leq 1$, since this is one of the coordinates of a point (x, y) in the parallelogram S_o . Thus we find ourselves again in the framework discussed in the preamble of the current proof.

First we note that if $\kappa_1 = 1$ or $\kappa_2 = 1$ then $\mu_1 = \mu_2$. Indeed, suppose $\kappa_1 = 1$. Then we have $\mu_1 - \mu_2 \geq 0$ and $(\mu_1 - \mu_2)(I - \Lambda) \leq (\mu_1 - \mu_2)(I - \Lambda(e_1 + e_2)) = 0$ from which we conclude that $\mu_1 = \mu_2$. As we have seen from Theorem 4.3, if

$$0 \leq \mu_1(I - \Lambda(e_1 + e_2)) = \mu_2(I - \Lambda(e_1 + e_2)) \leq 1$$

then ω is a q -weight. The same argument with the indices 1 and 2 interchanged shows that if $\kappa_2 = 1$ then ω is a range rank one q -weight.

Then to complete the proof of the theorem we may assume both $\kappa_1 < 1$ and $\kappa_2 < 1$. Let $X(t)$ be the matrix given by (53). Since $x_{12}(t) \leq \kappa_1(1 + x_{22}(t))$ and $x_{21}(t) \leq \kappa_2(1 + x_{11}(t))$ we have

$$\begin{aligned} \det(I + X(t)) &= (1 + x_{11}(t))(1 + x_{22}(t)) - x_{12}(t)x_{21}(t) \\ &\geq (1 + x_{11}(t))(1 + x_{22}(t))(1 - \kappa_1\kappa_2) \geq (1 - \kappa_1\kappa_2) \end{aligned}$$

so we can conclude the determinant is positive for all $t > 0$, hence the generalized boundary representation of ω is well-defined as discussed in the preamble. From equation (54) we conclude the generalized boundary representation is completely positive if and only if

$$(62) \quad \mu_1|_t \geq h_1(t)\mu_2|_t \quad \text{and} \quad \mu_2|_t \geq h_2(t)\mu_1|_t$$

for all $t > 0$. The conditions on μ_1 and μ_2 are $\mu_1 \geq \kappa_1\mu_2$, $\mu_2 \geq \kappa_2\mu_1$, $h_1(t) \leq \kappa_1 \leq 1$ and $h_2(t) \leq \kappa_2 \leq 1$ and these conditions ensure that (62) holds. Hence, we conclude that the generalized boundary representation is completely positive.

We recall that the proof that $h_1(t)$ and $h_2(t)$ are non increasing functions of t we only needed that these functions were bounded by one and the positivity of $\pi_t^\#$. Since these conditions are satisfied we know that these functions are non increasing and, therefore, $h_1(t) \rightarrow \kappa_1$ and $h_2(t) \rightarrow \kappa_2$ as $t \rightarrow 0+$. All that remains to show is that $\pi_t^\#$ is contractive and for this we only need show the first and second components of $\pi_t^\#(I)$ do not exceed one and as we have already calculated this condition will be met if and only if equations (60) and (61) hold for all $t > 0$. Now by the assumptions on μ_1 and μ_2 and the fact that we are in the case $\kappa_1 < 1$ and $\kappa_2 < 1$ all the terms above have limits and in the limit of $t \rightarrow 0+$ these inequalities become

$$0 \leq (\mu_1 - \kappa_1\mu_2)((I - \Lambda(e_1 + e_2))) \leq 1 - \kappa_1$$

and

$$0 \leq (\mu_2 - \kappa_2\mu_1)((I - \Lambda(e_1 + e_2))) \leq 1 - \kappa_2.$$

We know these inequalities are satisfied because these inequalities are precisely the inequalities that describe the parallelogram S_o and by the conditions of the theorem the point $x = \mu_1(I - \Lambda(e_1 + e_2))$ and $y = \mu_2(I - \Lambda(e_1 + e_2))$ lies in S_o . At first it may seem that knowing the inequalities are satisfied in the limit will not be of much help until we recall we have other inequalities at our disposal. Since $\mu_1 \geq \kappa_1\mu_2$ and $\mu_2 \geq \kappa_2\mu_1$ we have

$$(\mu_1 - \kappa_1\mu_2)((I - \Lambda(e_1 + e_2)) - (I - \Lambda(e_1 + e_2)))|_t \geq 0$$

$$(\mu_2 - \kappa_2\mu_1)((I - \Lambda(e_1 + e_2)) - (I - \Lambda(e_1 + e_2)))|_t \geq 0$$

$$(\mu_1 - \kappa_1\mu_2)((I - \Lambda(e_1 + e_2))|_t) \geq 0$$

$$(\mu_2 - \kappa_2\mu_1)((I - \Lambda(e_1 + e_2))|_t) \geq 0$$

and let us not forget the inequalities

$$h_1(t) \leq \kappa_1 < 1 \quad \text{and} \quad h_2(t) \leq \kappa_2 < 1$$

for all $t > 0$. To deal with these equations we will need to simplify notation. To this end let $x(t) = \mu_1((I - \Lambda(e_1 + e_2))|_t)$ and $y(t) = \mu_2((I - \Lambda(e_1 + e_2))|_t)$. Then $x(t)$ and $y(t)$ are non increasing functions of t and the inequalities we have shown to be true are

$$(63) \quad 0 \leq x(0) - \kappa_1 y(0) \leq 1 - \kappa_1$$

$$(64) \quad 0 \leq y(0) - \kappa_2 x(0) \leq 1 - \kappa_2$$

$$(65) \quad x(0) - \kappa_1 y(0) \geq x(t) - \kappa_1 y(t)$$

$$(66) \quad y(0) - \kappa_2 x(0) \geq y(t) - \kappa_2 x(t)$$

$$(67) \quad x(t) \geq \kappa_1 y(t)$$

$$(68) \quad y(t) \geq \kappa_2 x(t)$$

Now note that by equations (63), (65) and (67) we have that

$$0 \leq x(t) - \kappa_1 y(t) \leq 1 - \kappa_1.$$

Adding $(\kappa_1 - h_1(t))y(t)$ to both sides and using the fact that $y(t) \leq 1$ for all t , we obtain

$$(69) \quad 0 \leq x(t) - h_1(t)y(t) \leq 1 - h_1(t).$$

By using (64), (66) and (68) in the analogous way, we obtain:

$$0 \leq y(t) - h_2(t)x(t) \leq 1 - h_2(t).$$

These are precisely the desired inequalities. □

Theorem 6.3. *Let \mathfrak{H} be a separable Hilbert space. Suppose e_1 and e_2 are two positive operators in $\mathfrak{B}(\mathfrak{H})$ so that $0 \leq x_1 e_1 + x_2 e_2 \leq I$ if and only if both x_1 and x_2 lie in the closed interval $[0, 1]$. Let ω be a q -weight map of the form*

$$\omega(\rho)(A) = \rho(e_1)\mu_1(A) + \rho(e_2)\mu_2(A)$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{A}(\mathfrak{H})$ where μ_1 and μ_2 are positive boundary weights on $\mathfrak{A}(\mathfrak{H})$ satisfying the conditions given in Theorem 6.2 which ensure ω is a q -weight map. If ω is q -pure then $\mu_1 = \mu_2$ so ω has range rank one.

Proof. Assume the hypothesis of the theorem and suppose $\mu_1 \neq \mu_2$. Then the constants κ_1 and κ_2 defined in the previous theorem satisfy $\kappa_1 < 1$ and $\kappa_2 < 1$. Let us consider the boundary weight maps given by

$$\eta(\rho) = \rho(e_1)(\mu_1 - \kappa_1\mu_2) \quad \text{and} \quad \nu(\rho) = \rho(e_2)(\mu_2 - \kappa_2\mu_1)$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Let us show first that η is a q -weight, as the proof for ν is analogous. By Theorem 6.2, $\mu_1 - \kappa_1\mu_2$ is a positive boundary weight. Hence, by Theorem 4.3, since e_1 is a positive norm one operator, in order to obtain that η is a q -weight map, it suffices to show that

$$(\mu_1 - \kappa_1\mu_2)(I - \Lambda(e_1)) \leq 1.$$

Using the notation for the matrix $X(t)$ as in (53), and denoting $x(t) = \mu_1((I - \Lambda(e_1 + e_2))|_t)$, $y(t) = \mu_2((I - \Lambda(e_1 + e_2))|_t)$ we have that

$$\begin{aligned} (\mu_1 - \kappa_1\mu_2)((I - \Lambda(e_1))|_t) &= (\mu_1 - \kappa_1\mu_2)((I - \Lambda(e_1 + e_2) + \Lambda(e_2))|_t) \\ &= x(t) - \kappa_1 y(t) + x_{12}(t) - \kappa_1 x_{22}(t) \\ \text{by (50)} \quad &= x(t) - \kappa_1 y(t) + h_1(t)(1 + x_{22}(t)) - \kappa_1 x_{22}(t) \\ \text{by (69)} \quad &\leq 1 - \kappa_1 + h_1(t)(1 + x_{22}(t)) - \kappa_1 x_{22}(t) \\ &= 1 - (\kappa_1 - h_1(t))(1 + x_{22}(t)) \\ &\leq 1 \end{aligned}$$

Thus we have the desired inequality by taking the limit. By Theorem 4.3, the generalized boundary representation $\xi^\#$ for η is given by

$$\xi_t^\#(A) = \frac{(\mu_1 - \kappa_1\mu_2)(A|_t)}{1 + x_{11}(t) - \kappa_1 x_{21}(t)} e_1$$

Now let $\pi^\#$ be the generalized boundary representation for ω . Then, by Theorem 2.13, $\eta \leq_q \omega$ if and only if for all $t > 0$, $\pi_t^\# - \xi_t^\#$ is completely positive. Since e_1 and e_2 satisfy (48), in order to prove that $\pi_t^\# - \xi_t^\#$ is completely positive, it suffices to prove that

$$\ell_t^{(1)} - \frac{(\mu_1 - \kappa_1\mu_2)|_t}{1 + x_{11}(t) - \kappa_1 x_{21}(t)}$$

is positive (where $\ell_t^{(1)}$ is defined in (54)). In other words, we need to check that

$$\begin{aligned} ((1 + x_{11} - \kappa_1 x_{21})(1 + x_{22}) - \det(I + X))\mu_1|_t \\ + (\kappa_1 \det(I + X) - x_{12}(1 + x_{11} - \kappa_1 x_{21}))\mu_2|_t \geq 0. \end{aligned}$$

With some simplification this yields

$$x_{21}(x_{12} - \kappa_1(1 + x_{22}))\mu_1|_t + (\kappa_1(1 + x_{11} + x_{22} + x_{11}x_{22}) - x_{12}(1 + x_{11}))\mu_2|_t \geq 0.$$

Dividing by $(1 + x_{22})(1 + x_{11})$ we find this is equivalent to

$$h_2(t)(h_1(t) - \kappa_1)\mu_1|_t + (\kappa_1 - h_1(t))\mu_2|_t \geq 0.$$

And this inequality can be written in the form

$$(\kappa_1 - h_1(t))(\mu_2 - h_2(t)\mu_1)|_t \geq 0.$$

Now notice that this inequality holds because $\kappa_j \geq h_j(t)$ for $j = 1, 2$ and $\mu_2 \geq \kappa_2\mu_1$ by (51) and (52). Therefore $\eta \leq_q \omega$, and by an analogous argument with indices 1 and 2 exchanged we have that $\nu \leq_q \omega$. It is immediately apparent that neither $\eta \geq_q \nu$ nor $\nu \geq_q \eta$ so the subordinates of η are not well ordered. Hence, ω is not q -pure. \square

We remark that we do not know that all range rank two q -weights have the form assumed in the theorem, namely in terms of positive weights μ_1, μ_2 and positive operators e_1, e_2 such that $0 \leq x_1 e_1 + x_2 e_2 \leq I$ if and only if $x_1, x_2 \in [0, 1]$. However, at least in the interesting case when the q -weight is unital and has trivial normal spine, in other words it gives rise to an E_0 -semigroup of type II_0 , this is guaranteed by Proposition 6.1. Thus we immediately have the following result as an application of Theorem 6.3.

Corollary 6.4. *Let ω be a unital q -weight map over a separable Hilbert space \mathfrak{K} with range rank two. If its induced E_0 -semigroup has type II_0 then it is not q -pure.*

Theorem 6.5. *If ω is a q -weight map over \mathbb{C}^2 with trivial normal spine, then ω has range rank 1, 2, or 4. Furthermore, if ω is unital and q -pure, then it has range rank 1 or 4.*

Proof. Suppose that $\dim \text{Range}(\tilde{\omega}) > 2$, and let L be a boundary expectation for ω , so $\dim \text{Range}(L) = \dim \text{Range}(\tilde{\omega}) > 2$ by Theorem 3.2. Recall that the range \mathfrak{R}_L of L is a C^* -algebra under the norm and involution inherited from $M_2(\mathbb{C})$ but with Choi-Effros multiplication given by (17). Let \mathfrak{B} be the C^* -subalgebra of $M_2(\mathbb{C})$ generated by the range of L in the usual sense. Linearity of L and the definition of the multiplication operation in (17) imply that the restriction of L to \mathfrak{B} is a $*$ -homomorphism onto \mathfrak{R}_L .

Since $\dim \text{Range}(L) > 2$ and no C^* -subalgebra of $M_2(\mathbb{C})$ has dimension 3, it follows that $\mathfrak{B} = M_2(\mathbb{C})$, hence L is a $*$ -homomorphism from $M_2(\mathbb{C})$ onto \mathfrak{R}_L . Notice that $M_2(\mathbb{C})$ is simple, hence \mathfrak{R}_L is either $\{0\}$ or $M_2(\mathbb{C})$. Since $\tilde{\omega}$ is not the zero map, we have $\mathfrak{R}_L = M_2(\mathbb{C})$. Therefore, if ω is a q -weight map over \mathbb{C}^2 whose normal spine is zero, then ω has range rank 1, 2, or 4. To finish the proof, we note that by Corollary 6.4, there is no q -pure unital q -weight map with range rank 2 over \mathbb{C}^2 that has trivial normal spine. \square

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